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A New One Parameter Continuous Distribution using Three Component Distributions with Application to Real Life Data

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Abstract
A new one parameter lifetime distribution named "Fuyi distribution" for modeling lifetime data has been introduced. Some important mathematical properties of the proposed distribution including its shape, moments, skewness, kurtosis, coefficient of dispersion, index dispersion, quantile function, stochastic ordering, Bonferroni and Lorenz curves, moment generating function, characteristics function, distribution of ordered statistics, Renyi entropy measure, survival and hazard rate function, stress-strength reliability have been discussed. The estimation of its parameter has been discussed using maximum likelihood estimation. The usefulness and the applicability of the proposed distribution have been discussed and illustrated with two real lifetime data sets from medical science and environmental.

Keywords
Lifetime distribution, Moments, Hazard rate function, Bonferroni and Lorenz curves, Order statistics, Estimation of parameter, Goodness of fit

INTRODUCTION
Distribution theory and modelling plays a significant role in diverse fields such as; in medical science, engineering, economic and other science related areas. The need for the development of new distributions has become a necessity in this present ever evolving world, as different events are being carried out and these events generates data that needs to be studied for future insight.

Some data take shapes that may not be properly modelled by already existing probability distributions, thereby demanding the need for the modification/advancement of already existing probability distribution or the development of new probability distribution respectively. Another important key for modelling and analyzing data is the hazard rate or failure rate. Researchers such as Marshall, A.W. & Olkin, I. (2007) proposed a one parameter life time distribution having probability density function called the exponential distribution and it is used to model data with constant hazard rate function. To overcome the difficulty of the exponential distribution to model data with non-constant hazard rate, the following distributions were developed, for example; Ghitany M.E., Atieh B., & Nadarajah, S. (2008) proposed a new distribution by exploring a two-component distributions to obtain a one parameter distribution called the Lindley distribution with an increasing hazard rate function using the exponential distribution with scale parameter and a Gamma distribution having shape parameter 2 and scale parameter θ with mixing proportion $p=\frac{\theta}{\theta+1}$, Padgett W.J. (2011) proposed a two parameter distribution called the Weibull distribution used to model data with increasing hazard rate function, Rama Shanker (2016) proposed a one parameter distribution called the Aradhana distribution also using the three-component distribution by mixing the Exponential distribution with scale parameter θ, a Gamma distribution with shape and scale parameter 2 and θ, and a Gamma distribution with shape parameter 3 and scale parameter θ respectively with theirs mixing proportion of $\frac{\theta^2}{\theta^2+2\theta+2}, \frac{2\theta}{\theta^2+2\theta+2}, and\ \frac{2}{\theta^2+2\theta+2}$. Shukla KK. (2018) developed a one-parameter distribution called the Pranav distribution from two distributions namely; Exponential distribution with scale parameter θ and Gamma distribution having shape parameter 4 and scale parameter θ. Shanker R, and Shukla KK (2017) introduced a one-parameter distribution called the Ishita distribution based on a two-component mixture of the Exponential distribution having a scale parameter θ and a Gamma distribution having a shape parameter 3 and scale parameter θ with mixing proportion $p=\frac{\theta^3}{\theta^3+2}$, Kumar Devendra. (2017). introduced a two-parameter lifetime distribution named the Burr type XII distribution with some statistical properties, Lomax K.S. (1954) introduced a two-parameter distribution named Lomax distribution with an decreasing hazard rate function for modelling life time data, Rama Shanker (2015) studied a one-parameter distribution called the Akash distribution based on a two-component mixture of an Exponential distribution with scale parameter θ and a Gamma distribution having a shape parameter 2 and a scale parameter θ with a mixing proportion of $p=\frac{\theta}{\theta+1}$, other distribution introduced by mixing the exponential distribution and the gamma distribution are; the Rama distribution by Rama Shanker (2017), the Shanker distribution by Rama Shanker (2015), the Hamza distribution by Ahmad, Aijaz & Jallal, Muzamil & Ain, S Qurat & Tripathi, Rajnee. (2020).

There are some other existing distributions developed by transforming or modifying the already existing distributions, distributions such as; the Inverse and Inverse power Ishita by Shukla, K.K. (2021) and Frederick, A. O., Osuji, G.A., & Onyekwere, C.K. (2022) respectively, the Inverse and inverse power Hamza distribution by Okpala, I.F., Obiora-Ilouno, H.O., Omoruyi, F.A. (2023) and Omoruyi, Frederick & Chrisogonus, Onyekwere & Udofia, Edidiong & Florence, Ejiofor & Chukwunenye, Victor. (2023), the inverse power Rama by Chrisogonus K.O., George A.O., Samuel U.E. (2020), the new exponentiated Weibull distribution by Pal, Manish & Ali, M. & Woo, Jungsoo. (2003), a new variant of Rama distribution with simulation study and application to real life data by Omoruyi, F.A., Omeje, I.L., Anabike, I.C., & Obulezi, O.J. (2023), exponential inverse exponential by Pelumi Emmanuel Oguntunde, Adebowale Olusola Adejumo, and Ebahoro Alfred Owokolo (2017), the generalized inverted exponential distribution by Abouammoh, A.M. and Alshingiti, A.M. (2009), etc. In this paper, a new one-parameter continuous distribution having its probability density function (pdf)

$$f_{FD}(x,\theta)=\frac{\theta}{720(\theta^7+\theta^3+1)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x} \tag{1.1}$$

is proposed and we call this distribution, Fuyi Distribution. This distribution was developed due to the motivation of creating a distribution that will best fit a data during modelling, possibly the data may have already had other existing distributions that could already model it. But since the aim of every future event is to attain minimal error during forecasting, we hope to obtain a distribution that could perform better than some existing distribution, to obtain the mathematical properties of the proposed distribution.

The pdf (1.1) is a mixture of three distribution, Exponential distribution with scale parameter θ, Gamma distribution with shape and scale parameters 7 and θ respectively, and Gamma distribution with shape and scale parameter 3 and θ respectively. The mixture is of the form

$$f_{FD}(x,\theta)=p_1g_1(x;\theta)+p_2g_2(x;7,\theta)+p_3g_3(x;3,\theta)$$

where $p_1=\frac{1}{\theta^7+\theta^3+1}, p_2=\frac{\theta^7}{\theta^7+\theta^3+1}, and p_3=\frac{\theta^3}{\theta^7+\theta^3+1}$ are the mixing proportion such that, $p_1+p_2+p_3=1$. The cumulative density function is given as

$$F_{FD}(x,\theta)=1-\left[1+\frac{\theta^4x\left(\theta^9x^5+6\theta^8x^4+30\theta^7x^3+120\theta^6x^2+360\theta^5x+360\theta x+720\theta^4+720\right)}{720(\theta^7+\theta^3+1)}\right]e^{-\theta x} \tag{1.2}$$

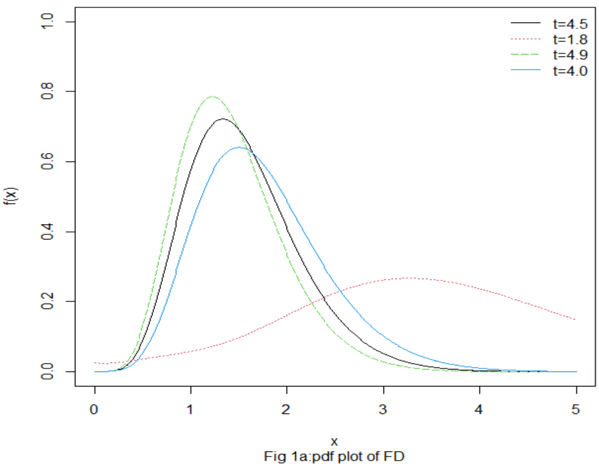


Fig. 1a: pdf plot of $X \sim FD(x,\theta)$

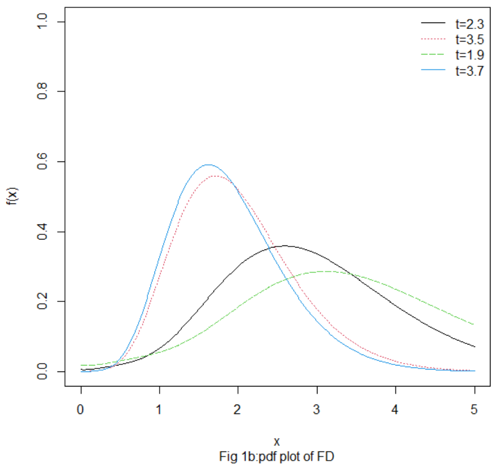


Fig. 1b: pdf plot of $X \sim FD(x,\theta)$

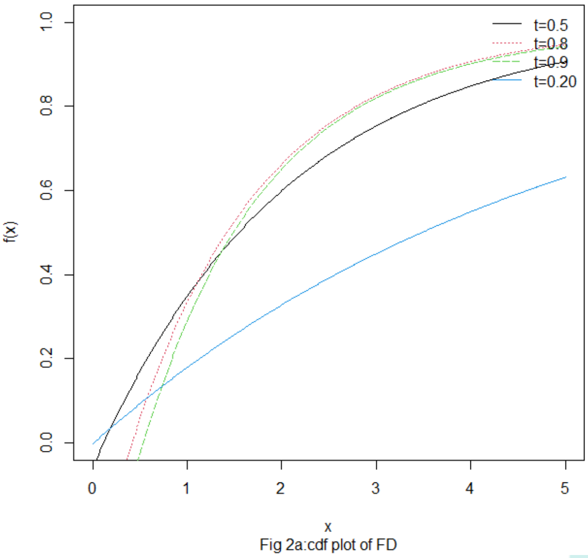


Fig. 2a: cdf plot of $X \sim FD(x,\theta)$

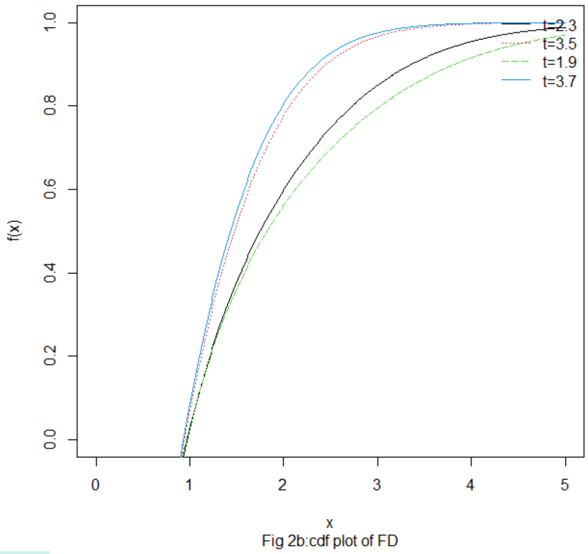


Fig. 2b: cdf plot of $X \sim FD(x,\theta)$

STATISTICAL PROPERTIES OF FUYI DISTRIBUTION

Moment

The rth non-central moment of a Fuyi random variable X is given as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_{FD}(x)dx = \int_0^\infty x^r \frac{\theta}{720(\theta^7+\theta^3+1)}(\theta^{13}x^6+360\theta^5x^2+720)e^{-\theta x}dx$$

$$E(X^r) = \frac{\theta^7\Gamma(r+7)+360\theta^3\Gamma(r+3)+720\Gamma(r+1)}{720\theta^r(\theta^7+\theta^3+1)} \quad (2.1)$$

Mean

The arithmetic mean is obtained from equation (2.1) above by substituting r = 1

$$\mu = \frac{7\theta^7+3\theta^3+1}{\theta(\theta^7+\theta^3+1)} \quad (2.2)$$

Other useful non-central moments

The 2nd, 3rd and 4th non-central moment are obtained from equation (2.1) by substituting r = 2, r = 3, and r = 4 respectively

$$\mu_2' = \frac{2(28\theta^7+6\theta^3+1)}{\theta^2(\theta^7+\theta^3+1)} \quad (2.3)$$

$$\mu_3' = \frac{6\left(84\theta^7 + 10\theta^3 + 1\right)}{\theta^3\left(\theta^7 + \theta^3 + 1\right)} \quad (2.4)$$

$$\mu_4' = \frac{24\left(210\theta^7 + 15\theta^3 + 1\right)}{\theta^4\left(\theta^7 + \theta^3 + 1\right)} \quad (2.5)$$

Useful central moments

The 2nd, 3rd and 4th central moments are respectively

$$\sigma^2 = \mu_2' - \mu^2 \quad (2.6)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 \quad (2.7)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4 \quad (2.8)$$

$$\sigma^2 = \frac{105\theta^{14} + 110\theta^{10} + 72\theta^7 + 21\theta^6 + 20\theta^3 + 3}{\theta^2\left(\theta^7 + \theta^3 + 1\right)^2} \quad (2.9)$$

$$\mu_3 = \frac{14\theta^{21} + 18\theta^{17} - 78\theta^{14} - 366\theta^{13} + 372\theta^{10} + 6\theta^9 + 342\theta^7 + 522\theta^6 + 30\theta^3 + 2}{\theta^3\left(\theta^7 + \theta^3 + 1\right)^3} \quad (2.10)$$

$$\mu_4 = \frac{\begin{array}{l} -4293\theta^{28} - 126698\theta^{24} - 115010\theta^{21} - 132300\theta^{20} - 251770\theta^{17} - 4358\theta^{16} - 128190\theta^{14} \\ -110350\theta^{13} - 285\theta^{12} - 88160\theta^{10} - 851\theta^9 - 1707\theta^7 - 789\theta^6 - 225\theta^3 - 14 \end{array}}{\theta^4\left(\theta^7 + \theta^3 + 1\right)^4} \quad (2.11)$$

Coefficient of skewness

The coefficient of skewness of Fuyi distribution is given as

$$\gamma = \frac{\mu_3}{\left(\sigma^2\right)^{\frac{3}{2}}} = \frac{14\theta^{21} + 18\theta^{17} - 78\theta^{14} - 366\theta^{13} + 372\theta^{10} + 6\theta^9 + 342\theta^7 + 522\theta^6 + 30\theta^3 + 2}{\left(105\theta^{14} + 110\theta^{10} + 72\theta^7 + 21\theta^6 + 20\theta^3 + 3\right)^{\frac{3}{2}}} \quad (2.12)$$

Coefficient of kurtosis

The coefficient of kurtosis of Fuyi distribution is given as

$$\beta = \frac{\mu_4}{\left(\sigma^2\right)^2} = \frac{\begin{array}{l} -4293\theta^{28} - 126698\theta^{24} - 115010\theta^{21} - 132300\theta^{20} - 251770\theta^{17} - 4358\theta^{16} - 128190\theta^{14} \\ -110350\theta^{13} - 285\theta^{12} - 88160\theta^{10} - 851\theta^9 - 1707\theta^7 - 789\theta^6 - 225\theta^3 - 14 \end{array}}{\left(105\theta^{14} + 110\theta^{10} + 72\theta^7 + 21\theta^6 + 20\theta^3 + 3\right)^2} \quad (2.13)$$

Coefficient of variation

The coefficient of variation of Fuyi distribution is given as

$$\xi = \frac{\sigma}{\mu} \times \frac{100}{1} = \frac{100\sqrt{105\theta^{14} + 110\theta^{10} + 72\theta^7 + 21\theta^6 + 20\theta^3 + 3}}{7\theta^7 + 3\theta^3 + 1} \quad (2.14)$$

Index of dispersion

The index of dispersion of Fuyi distribution is given as

$$\eta = \frac{\sigma^2}{\mu_1'} = \frac{105\theta^{14} + 110\theta^{10} + 72\theta^7 + 21\theta^6 + 20\theta^3 + 3}{\theta\left(7\theta^7 + 3\theta^3 + 1\right)\left(\theta^7 + \theta^3 + 1\right)} \quad (2.15)$$

Quantile Function

The q quantile of Fuyi distribution is obtained using $F\left(x_q\right) = P\left(X \le x_q\right) = q$ for $0 < q < 1$. Replace x with x_q in the cdf of Fuyi distribution and equate to q

$$q = 1 - \left[1 + \frac{\theta^4 x_q\left(\theta^9 x_q^{\,5} + 6\theta^8 x_q^{\,4} + 30\theta^7 x_q^{\,3} + 120\theta^6 x_q^{\,2} + 360\theta^5 x_q + 360\theta x_q + 720\theta^4 + 720\right)}{720\left(\theta^7 + \theta^3 + 1\right)}\right] e^{-\theta x_q}$$

$$\left(1 - q\right)\left(720\left(\theta^7 + \theta^3 + 1\right)\right) = \left(\begin{array}{l} 720\left(\theta^7 + \theta^3 + 1\right) + \theta^4 x_q \\ \left(\theta^9 x_q^5 + 6\theta^8 x_q^4 + 30\theta^7 x_q^3 + 120\theta^6 x_q^2 + 360\theta^5 x_q + 360\theta x_q + 720\theta^4 + 720\right) \end{array}\right) e^{-\theta x_q} \quad (2.16)$$

Stochastic Ordering of Fuyi Distribution

Let $X \sim F\left(\theta_1\right)$ and $Y \sim F\left(\theta_2\right)$. If $\theta_1 > \theta_2$, then $X \le_{lr} Y$ hence $X \le_{hr} Y$, $X \le_{mlr} Y$, and $X \le_{st} Y$

Proof:

$$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{\theta_1}{720\left(\theta_1^7+\theta_1^3+1\right)}\left(\theta_1^{13}x^6+360\theta_1^5x^2+720\right)e^{-\theta_1 x}}{\frac{\theta_{21}}{720\left(\theta_2^7+\theta_2^3+1\right)}\left(\theta_2^{13}x^6+360\theta_2^5x^2+720\right)e^{-\theta_2 x}}$$

$$=\left[\frac{\theta_1\left(\theta_2^7+\theta_2^3+1\right)}{\theta_2\left(\theta_1^7+\theta_1^3+1\right)}\right]\left[\frac{\theta_1^{13}x^6+360\theta_1^5x^2+720}{\theta_2^{13}x^6+360\theta_2^5x^2+720}\right]e^{-x(\theta_2-\theta_1)} \tag{2.17}$$

Taking natural log of eqn. 2.17, will yield

$$\ln\left[\frac{f_X(x)}{f_Y(x)}\right]=\ln\left[\frac{\theta_1\left(\theta_2^7+\theta_2^3+1\right)}{\theta_2\left(\theta_1^7+\theta_1^3+1\right)}\right]+\ln\left[\frac{\theta_1^{13}x^6+360\theta_1^5x^2+720}{\theta_2^{13}x^6+360\theta_2^5x^2+720}\right]+(\theta_2-\theta_1)x$$

Differentiating the natural log of eqn. 2.17, will yield

$$\frac{d}{dx}\left[\ln\left(\frac{f_X(x)}{f_Y(x)}\right)\right]=\frac{1440\left(\theta_1^{13}\theta_2^5-\theta_1^5\theta_2^{13}\right)x^7+4320x^5\left(\theta_1^{13}-\theta_2^{13}\right)+518400x\left(\theta_1^5-\theta_2^5\right)}{\left(\theta_1^{13}x^6+360\theta_1^5x^2+720\right)\left(\theta_2^{13}x^6+360\theta_2^5x^2+720\right)}+(\theta_2-\theta_1)=0$$

Bonferonni and Lorenz Curve

The Bonferonni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{1}{p\mu}\left[\int_0^\infty xf(x)dx-\int_q^\infty xf(x)dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty xf(x)dx\right] \tag{2.18}$$

$$B(p)=\frac{720(7\theta^8+3\theta^4+\theta)-\theta^8\gamma_{(8,q)}-360\theta^3\gamma_{(4,q)}-720\gamma_{(2,q)}}{720p(7\theta^8+3\theta^4+\theta)} \tag{2.19}$$

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{1}{\mu}\left[\int_0^\infty xf(x)dx-\int_q^\infty xf(x)dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty xf(x)dx\right] \tag{2.20}$$

$$L(p)=\frac{720(7\theta^8+3\theta^4+\theta)-\theta^8\gamma_{(8,q)}-360\theta^3\gamma_{(4,q)}-720\gamma_{(2,q)}}{720(7\theta^8+3\theta^4+\theta)} \tag{2.21}$$

Maximum likelihood estimation of the Fuyi distribution parameter

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample drawn from a Fuyi distribution, then the likelihood function is given as

$$\ell\left(f_{FD}(x,\theta)\right)=\prod_{i=1}^{n}\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}$$

$$=\left(\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\right)^n e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(\theta^{13}x^6+360\theta^5x^2+720\right) \tag{2.22}$$

Taking natural log of eqn. 2.22 will yield

$$\text{Ln}(\ell\left(f_{FD}(x,\theta)\right))=n\ln\left(\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\right)-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln\left(\theta^{13}x^6+360\theta^5x^2+720\right)$$

Differentiating the natural log of eqn.2.22 will yield

$$=\frac{n\left(1-6\theta^7-2\theta^3\right)}{\theta\left(\theta^7+\theta^3+1\right)}-\sum_{i=1}^{n}x_i+\frac{13\theta^{12}x^6+1800\theta^4x^2}{\theta^{13}x^6+360\theta^5x^2+720}=0 \tag{2.23}$$

The MLE is implemented using Newton-Raphson’s numerical iterative method since it has no closed-form solution.

Moment generating function of Fuyi distribution

The moment generating function of a X ~ Fuyi (θ) is given by

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f_{FD}(x)dx \tag{2.24}$$

$$=\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\int_0^\infty e^{tx}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}dx \tag{2.25}$$

$$= \frac{\theta}{720\left(\theta^7+\theta^3+1\right)} \int_0^\infty e^{-x(\theta-t)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)dx$$

$$= \frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left[\theta^{13}\int_0^\infty x^{7-1}e^{-x(\theta-t)}dx+360\theta^5\int_0^\infty x^{3-1}e^{-x(\theta-t)}dx+720\int x^{1-1}e^{-x(\theta-t)}dx\right]$$

where,

$$\int_0^\infty x^{\alpha-1}e^{-\theta x}dx = \frac{\Gamma(\alpha)}{\theta^\alpha} \tag{2.26}$$

We have that;

$$= \frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left[\theta^{13}\frac{\Gamma 7}{(\theta-t)^7}+360\theta^5\frac{\Gamma 3}{(\theta-t)^3}+720\frac{\Gamma 1}{(\theta-t)}\right]$$

$$M_X(t)=\frac{\theta^{14}+\theta^6(\theta-t)^4+\theta(\theta-t)^6}{(\theta-t)^7\left(\theta^7+\theta^3+1\right)} \tag{2.27}$$

Characteristic function of Fuyi distribution

The moment generating function of a X ~ Fuyi (θ) is given by

$$\phi_X(t)=E\left(e^{-itx}\right)=\int_0^\infty e^{-itx}f_{FD}(x)dx \tag{2.28}$$

$$\phi_X(t)=\frac{\theta^{14}+\theta^6(\theta-it)^4+\theta(\theta-it)^6}{(\theta-it)^7\left(\theta^7+\theta^3+1\right)} \tag{2.29}$$

Distribution of the order statistics

Suppose $x_1, x_2, x_3, ..., x_n$ is a random sample of $X_{(r)}; r=1,2,3,...,n$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude ∋ $X_1 \le X_2 \le X_3 \le ... \le X_r$ and $X_1=\min(X_1, X_2, X_3, ..., X_r)$, $X_n=\max(X_1, X_2, X_3, ..., X_r)$ then the probability density function of the rth order statistics is given by

$$f_{r:n}(x;\theta)=\frac{n!}{(r-1)!(n-r)!}f_{FD}(x;\theta)\left[F_{FD}(x;\theta)\right]^{r-1}\left[1-F_{FD}(x;\theta)\right]^{n-r} \tag{2.30}$$

where $f(.)$ and $F(.)$ are the pdf and cdf of Chris-Jerry distribution respectively. Hence, we have

$$f_{r:n}(x;\theta)=\frac{n!\theta\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}}{(r-1)!(n-r)!720\left(\theta^7+\theta^3+1\right)}\left[1-\left[1+\frac{\theta^4x_q\left(\theta^9x_q{}^5+6\theta^8x_q{}^4+30\theta^7x_q{}^3+120\theta^6x_q{}^2+360\theta^5x_q+360\theta x_q+720\theta^4+720\right)}{720\left(\theta^7+\theta^3+1\right)}\right]e^{-\theta x_q}\right]^{r-1} \tag{2.31}$$

$$\left[\left[1+\frac{\theta^4x_q\left(\theta^9x_q{}^5+6\theta^8x_q{}^4+30\theta^7x_q{}^3+120\theta^6x_q{}^2+360\theta^5x_q+360\theta x_q+720\theta^4+720\right)}{720\left(\theta^7+\theta^3+1\right)}\right]e^{-\theta x_q}\right]^{n-r}$$

The pdf of the largest order statistics is obtained by setting $r=n$

$$f_{n:n}(x;\theta)=\frac{n\theta\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}}{720\left(\theta^7+\theta^3+1\right)}\left[1-\left[1+\frac{\theta^4x_q\left(\theta^9x_q{}^5+6\theta^8x_q{}^4+30\theta^7x_q{}^3+120\theta^6x_q{}^2+360\theta^5x_q+360\theta x_q+720\theta^4+720\right)}{720\left(\theta^7+\theta^3+1\right)}\right]e^{-\theta x_q}\right]^{n-1}$$

$$f_{1:n}(x;\theta)=\frac{n\theta\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}}{720\left(\theta^7+\theta^3+1\right)}\left[\left[1+\frac{\theta^4x_q\left(\theta^9x_q{}^5+6\theta^8x_q{}^4+30\theta^7x_q{}^3+120\theta^6x_q{}^2+360\theta^5x_q+360\theta x_q+720\theta^4+720\right)}{720\left(\theta^7+\theta^3+1\right)}\right]e^{-\theta x_q}\right]^{n-1}$$

Information measure and asymptotic behaviour of Fuyi distribution

Entropy is the quantity of uncertainty or randomness in a system. It is an information measure for non-negative $\omega \neq 1$.

The Rényi Entropy for Fuyi distributed random variable X is

$$R_\omega(x)=\lim_{n\to\infty}\left(I_\omega(f_n)-\log n\right)=\frac{1}{1-\omega}\log\int_0^\infty f^\omega(x)dx \tag{2.32}$$

For ω → 1, we have the special case of Shannon Entropy $R_S(x)$.

$$= \frac{1}{1-\omega}\log\int_0^\infty\left(\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x}\right)^\omega dx$$

$$= \sum_{k=0}^{\omega}\sum_{l=0}^{k}\binom{\omega}{k}\binom{k}{l}\frac{\theta^{\omega+5k+8l}}{2^k360^l\left(\theta^7+\theta^3+1\right)}\int_0^\infty x^{(2k+4l+1)}e^{-\omega\theta x}dx$$

$$= \sum_{k=0}^{\omega}\sum_{l=0}^{k}\binom{\omega}{k}\binom{k}{l}\frac{\theta^{\omega+5k+8l}\Gamma\left(2k+4l+1\right)}{2^k360^l\left(\theta^7+\theta^3+1\right)\left(\theta\omega\right)^{2k+4l+1}}$$

$$= \sum_{k=0}^{\omega}\sum_{l=0}^{k}\binom{\omega}{k}\binom{k}{l}\frac{\theta^{\omega+3k+4l-1}\Gamma\left(2k+4l+1\right)}{2^k360^l\left(\theta^7+\theta^3+1\right)\left(\omega\right)^{2k+4l+1}} \qquad (2.33)$$

The asymptotic behaviour of the Fuyi distributed random variable is investigated by taking the limit of the pdf as $x \to 0$ and as $x \to \infty$.

$$\lim_{x\to 0}\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x} = \frac{\theta}{\left(\theta^7+\theta^3+1\right)} \qquad (2.34)$$

and

$$\lim_{x\to\infty}\frac{\theta}{720\left(\theta^7+\theta^3+1\right)}\left(\theta^{13}x^6+360\theta^5x^2+720\right)e^{-\theta x} = 0 \qquad (2.35)$$

Survival function and failure rate

Given a continuous distribution with pdf and cdf in equations (1.1) and (1.2), the survival function is given by

$$S_{FD}(x;\theta) = 1 - F_{FD}(x;\theta) = \left[1+\frac{\theta^4x\left(\theta^9x^5+6\theta^8x^4+30\theta^7x^3+120\theta^6x^2+360\theta^5x+360\theta x+720\theta^4+720\right)}{720\left(\theta^7+\theta^3+1\right)}\right]e^{-\theta x_q},\ x,\theta>0 \qquad (2.36)$$

Notice that for Fuyi distribution the survival function $S_{FD}(x;\theta) = 1$ as x $\to$ 0 and $S_{FD}(x;\theta) = 0$ as x $\to\infty$. Also, the failure rate $h_{FD}(x;\theta)$, an important tool in reliability measure and engineering is given by

$$h_{FD}(x,\theta) = \frac{f_{FD}(x,\theta)}{S_{FD}(x,\theta)} = \frac{\theta\left(\theta^{13}x^6+360\theta^5x^2+720\right)}{720\left(\theta^7+\theta^3+1\right)+\theta^4x\left(\theta^9x^5+6\theta^8x^4+30\theta^7x^3+120\theta^6x^2+360\theta^5x+360\theta x+720\theta^4+720\right)} \qquad (2.37)$$

For Fuyi distribution, the failure rate exhibits the following behavior; $h_{FD}(x,\theta) = \frac{\theta}{\left(\theta^7+\theta^3+1\right)}$ as x $\to$ 0 and $h_{FD}(x,\theta) = 0$ as x $\to\ \infty$.

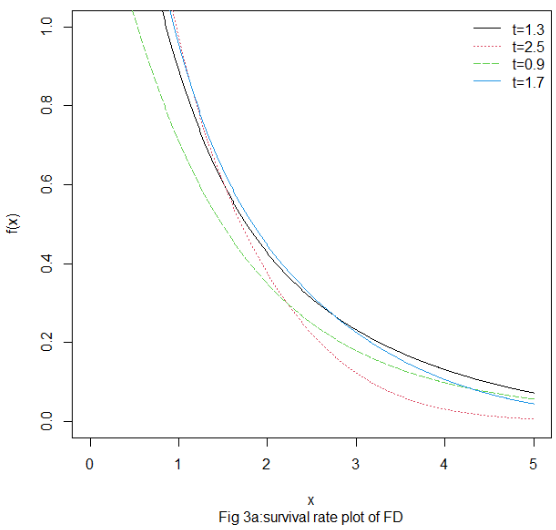


Fig. 3a: survival rate plot of $X \sim FD(x,\theta)$

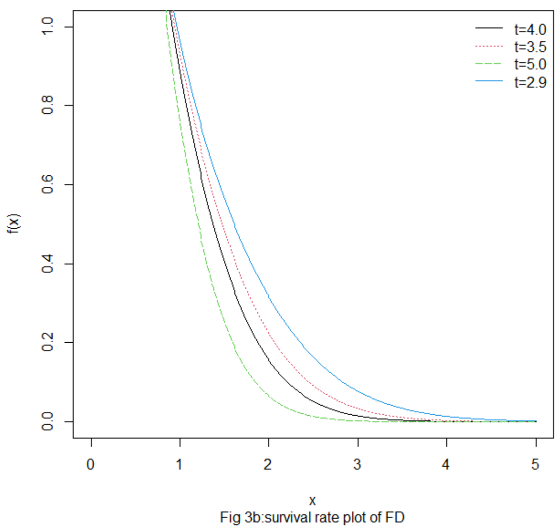
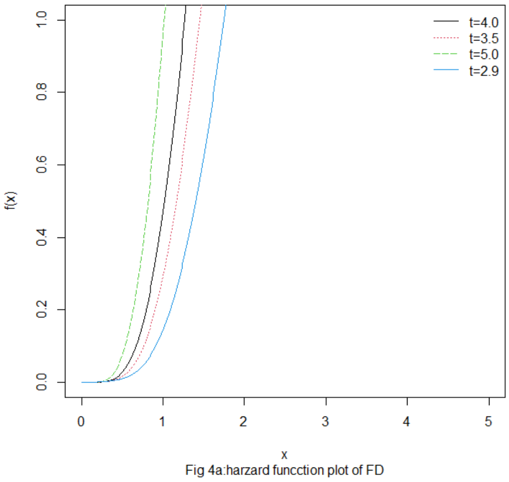


Fig. 3b: survival rate plot of $X \sim FD(x,\theta)$



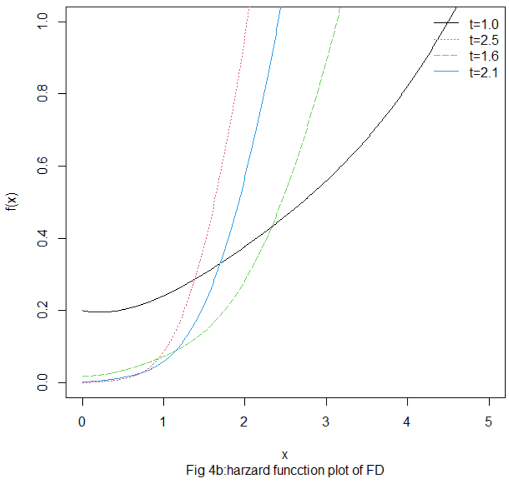


Fig. 4a: hazard function plot of $X \sim FD(x, \theta)$ **Fig. 4b:** hazard function plot of $X \sim FD(x, \theta)$

The Figs. 3a and 3b show the plots of survival function and figs 4a and 4b are the plots of the hazard function for various parameter values.

Stress-Strength reliability

An examination of the Stress-Strength Reliability of Fuyi distribution is also carried out. The stress-strength reliability is used to measure the life of a component that possesses random strength X and subjected to random stress Y. In a case, where the applied stress Y is higher than the strength x of the system, that is X < Y, the component fails. For the component to function efficiently, the strength of the system must be greater than the stress applied to it. Hence, R = P (Y< X) is the measure of the reliability of a component and find application in aging of concrete pressure vessels deteriorating of rocket motors, ceramic components and so on.

Given that X and Y are independent random variables denoting strength and stress of a component. We assume further that X and Y follow Fuyi distribution with pdf given in equation (1.1), with parameter θ_1 and θ_2 respectively. Then, the stress-strength reliability is obtained as follows:

$$R = P(Y < X) = \int_0^\infty P(Y < X = x) f_X(x)\, dx = \int_0^\infty f_X(x, \theta_1) F(x, \theta_2) dx$$

Therefore, we have;

$$R = \int_0^\infty \left[\frac{\theta\left(\theta_1^{13} x^6 + 360\theta_1^5 x^2 + 720\right)}{720(\theta_1^7 + \theta_1^3 + 1)} e^{-\theta_1 x} \right] \left[1 - \left\{ 1 + \frac{\left(\theta_2^4 x(\theta_2^9 x^5 + 6\theta_2^8 x^4 + 30\theta_2^7 x^3 + 120\theta_2^6 x^2 + 360\theta_2^5 x + 360\theta_2 x + 720\theta_2^4 + 720\right)}{720(\theta_2^7 + \theta_2^3 + 1)} \right\} e^{-\theta_2 x} \right] \quad (2.38)$$

$$R = 1 - \frac{\left\{ \begin{array}{c} (\theta_1\theta_2^7+\theta_1\theta_2^3+\theta_1)\theta_1^{13}(\theta_1+\theta_2)^6+(\theta_1\theta_2^7+\theta_1\theta_2^3+\theta_1)\theta_1^5(\theta_1+\theta_2)^{10} \\ +(\theta_1\theta_2^7+\theta_1\theta_2^3+\theta_1)(\theta_1+\theta_2)^{12}+924\theta_1^{14}\theta_2^{13}+28(\theta_1\theta_2^{13}+\theta_1^{14}\theta_2^5+\theta_1^{14}\theta_2^9)(\theta_1+\theta_2)^4+ \\ 462\theta_1^{14}\theta_2^{12}(\theta_1+\theta_2)+210\theta_1^{14}\theta_2^{11}(\theta_1+\theta_2)^2+84\theta_1^{14}\theta_2^{10}(\theta_1+\theta_2)^3 \\ +7(3\theta_1\theta_2^{12}+\theta_1^{14}\theta_2^8+\theta_1^{14}\theta_2^4)(\theta_1+\theta_2)^5+(\theta_1\theta_2+15\theta_1\theta_2^{11})(\theta_1+\theta_2)^6 \\ +(\theta_1\theta_2^{12}+10\theta_1\theta_2^{10})(\theta_1+\theta_2)^7+(\theta_1\theta_2^{11}+6\theta_1\theta_2^9+6\theta_1\theta_2^5)(\theta_1+\theta_2)^8+ \\ (\theta_1\theta_2^{10}+3\theta_1\theta_2^8+3\theta_1\theta_2^4)(\theta_1+\theta_2)^9+(\theta_1\theta_2^9+\theta_1\theta_2^5)(\theta_1+\theta_2)^{10}+ \\ (\theta_1\theta_2^8+\theta_1\theta_2^4)(\theta_1+\theta_2)^{11} \end{array} \right\}}{(\theta_1+\theta_2)^{13}(\theta_1^7+\theta_1^3+1)(\theta_2^7+\theta_2^3+1)} \quad (2.39)$$

Applications to Lifetime Data

Fuyi distribution has been fitted to some real lifetime data sets and it gives better fit than Log Normal, Burr XII, Weibull, Gamma, Lomax, New Exponentiated Weibull distribution, Extended Inverse Exponential, and Generalized inverted exponential distributions.

Table 1 List of one-parameter distributions For Comparisom

Distribution	$f(x)$	$F(x)$
Proposed Fuyi Distribution	$\dfrac{\theta(\theta^{13}x^6+360\theta^5x^2+720)e^{-\theta x}}{720(\theta^7+\theta^3+1)}$	1- $\left[1+\dfrac{\theta^4x\left(\begin{array}{c}\theta^9x^5+6\theta^8x^4+30\theta^7x^3+120\theta^6x^2+360\theta^5x\\+360\theta x+720\theta^4+720\end{array}\right)}{720(\theta^7+\theta^3+1)}\right]$
Burr XII Distribution	$ck\dfrac{x^{c-1}}{(1+x^c)^{k+1}}$	$1-(1+x^c)^{-k}$
Lnormal Distribution	$\dfrac{1}{x\sigma\sqrt{2\pi}}exp\left(-\dfrac{(Inx-\mu)^2}{2\sigma^2}\right)$	$\dfrac{1}{2}\left[1+erf\left(\dfrac{Inx-\mu}{\sigma\sqrt{2}}\right)\right]$
GIE Distribution	$\dfrac{\alpha\lambda}{x^2}\left(exp\left[-\dfrac{\lambda}{x}\right]\right)\left(1-exp\left[-\dfrac{\lambda}{x}\right]\right)^{\alpha-1}$	$1-\left(1-exp\left[-\dfrac{\lambda}{x}\right]^{\alpha}\right)$
Lomax Distribution	$\dfrac{\alpha\left(1+{x}/{\lambda}\right)^{-(\alpha+1)}}{\lambda}$	$1-\left(1+{x}/{\lambda}\right)^{-(\alpha+1)}$
Gamma Distriibution	$\dfrac{\theta^{\alpha}x^{\alpha-1}}{\Gamma(\alpha)}e^{-\theta x}$	$\dfrac{1}{\Gamma(\alpha)}\gamma(\alpha,\theta x)$
Weibull Distribution	$\dfrac{k}{\lambda}\left(\dfrac{x}{\lambda}\right)^{k-1}e^{-\left({x}/{\lambda}\right)^k}$	$1-e^{-\left({x}/{\lambda}\right)^k}$
Extended Inverse Exponential Distribution	$\left(\dfrac{\alpha\theta}{x^2}\right)a\left(\dfrac{1}{(1-a)^2}\right)exp\big(-\alpha(a/(1-a))\big)$ Where, $a=exp(-\theta/x)$	$1-exp\big(-\alpha(a/(1-a))\big)$ Where, $a=exp(-\theta/x)$
NEX Distribution	$\alpha\gamma\lambda^{\gamma}x^{\gamma-1}[1-a]^{\alpha-1}a$ Where, $a=exp\{-(\lambda x)^{\gamma}\}$	$[1-exp\{-(\lambda x)^{\gamma}\}]^{\alpha}$

The application is on rainfall reported at Los Angeles Civic center from1943 to 2018 and studied by Mustapha nadar and faith kizilaslan (2015), the data is in table 1

Table 2 The Rainfall reported at the Los Angeles Civic Centre from 1943 to 2018 in the month on March

4.55	2.47	3.43	3.66	0.79	3.07	1.40	0.87	0.44	6.14	0.48	2.99	0.56	1.02
5.30	0.31	0.57	1.10	2.78	1.79	2.49	0.53	2.50	3.34	1.49	2.36	0.53	2.07
3.78	4.83	1.81	1.89	8.02	5.85	4.79	4.10	3.54	8.37	0.28	1.29	5.27	0.95
0.26	0.81	0.17	5.92	7.12	2.74	1.86	6.98	2.16	4.06	1.24	2.82	1.17	0.32
4.32	1.47	2.14	2.87	0.05	0.01	0.35	0.48	3.96	1.75	0.54	1.18	0.87	1.60
0.09	2.69												

The determiners of model performance used here are the Akaike Information Criterion, Corrected Akaike Information Criterion, Bayesian Information Criterion, Hannan-Quinn Information Criterion, negative Log-Likelihood, Cramer Von Mises (W*), Anderson Darling (A*), While the Kolmogorov-Smirnov (K-S) statistic and the p-value determines the fitness of the distribution to the data.

Table 3 Analytical measures of performance and fitness using the Rainfall reported Data at Los Angeles

Distr.	**NLL**	**AIC**	**CAIC**	**BIC**	**HQIC**	W^*	A^*	θ	β	**K-S**	**P-value**
Fuyi	-137.42	276.836	276.893	279.112	277.742	0.047	0.302	0.4484	-	0.0650	0.9212
Burr XII	-146.66	297.326	297.499	301.879	299.139	0.271	0.631	1.5858	0.7163	0.1418	0.1108
NEX	-137.08	278.151	278.325	282.704	279.963	0.057	0.353	0.1774	1.2550	0.1844	0.0150
EIE	-140.00	276.940	277.114	281.493	278.753	0.037	0.245	0.0034	0.0083	0.0795	0.7533
Weibull	-135.87	275.816	275.991	280.370	277.630	0.029	0.187	1.1183	2.5477	0.0721	0.8485
Gamma	-136.17	276.347	276.521	280.900	278.160	0.036	0.234	1.1373	2.1543	0.0766	0.7914
Lomax	-136.53	277.061	277.235	281.615	278.874	0.037	0.236	30111970	12287832	0.0794	0.7534
GIE	-183.70	371.401	371.575	375.955	373.214	1.146	6.656	0.4521	0.1898	0.2663	7.347e-05
Lnormal	-145.70	296.367	296.541	300.921	298.180	0.227	1.444	0.3756	1.3388	0.1102	0.3461

Based on the results in table 3, the proposed Fuyi distribution fits the rainfall reported data at Los Angeles Civic Centre having the highest p-value= 0.9212. However, in parameter estimation it performed better than all competing distributions except the Weibull distribution. The performance measure for the proposed distribution, Fuyi distribution, had the least information criterions among others, except for that of the Weibull distribution.

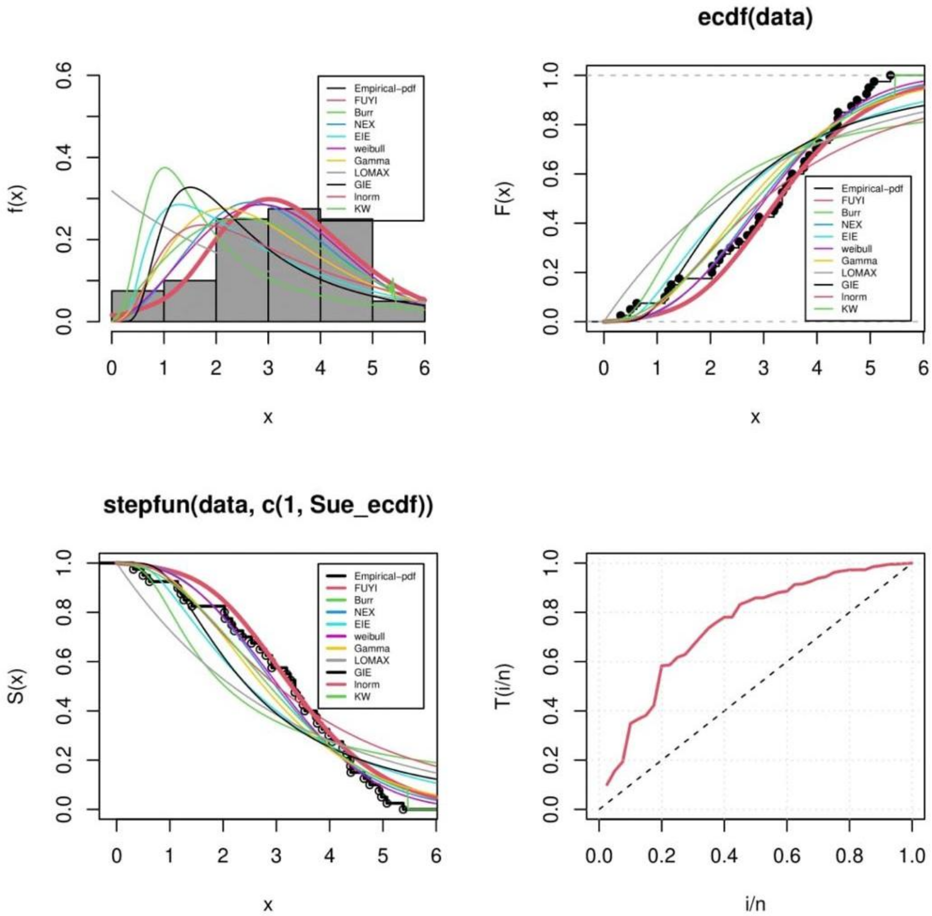


Fig. 5 Density, cdf, survival, and TTT plots for the Rainfall reported at the Los Angeles Civic Centre from 1943 to 2018 in the month of March

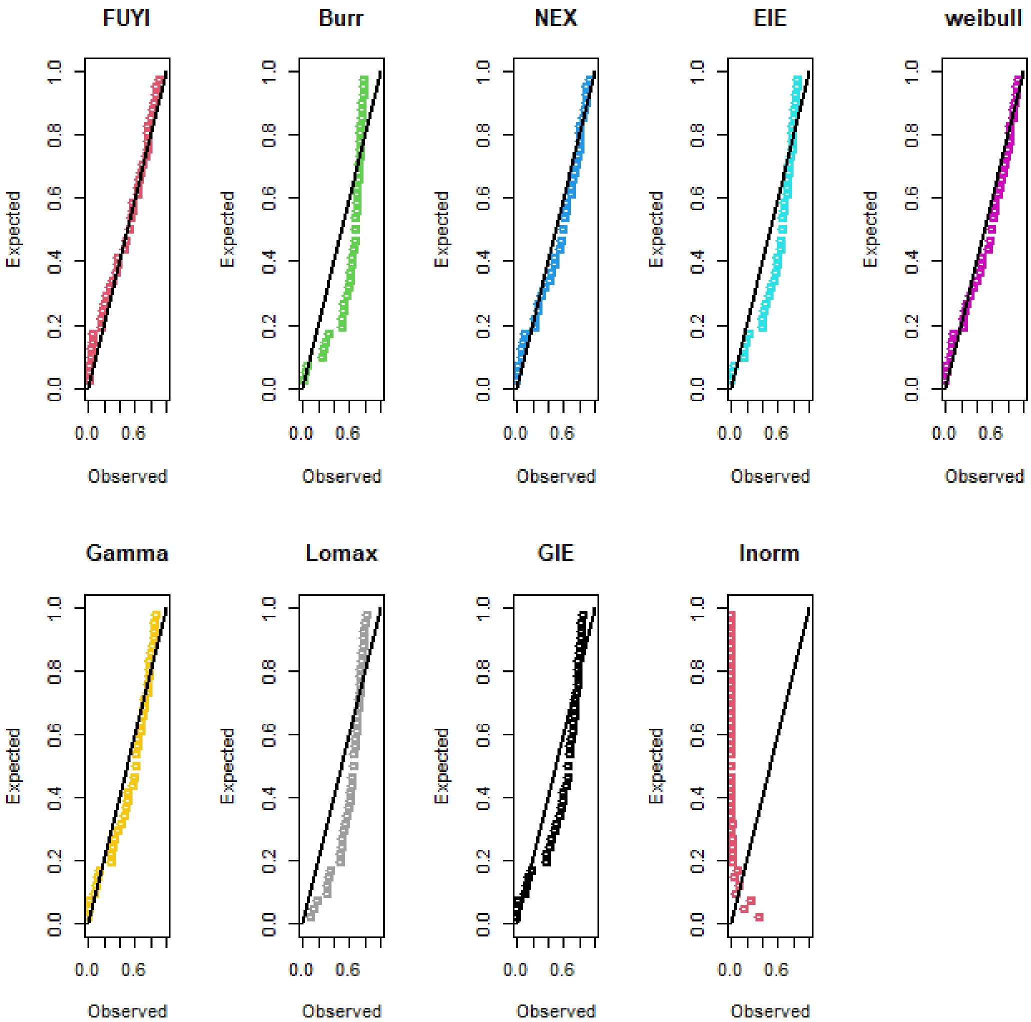


Fig. 6 PP plots for the Rainfall Data reported at the Los Angeles Civic Centre from 1943 to 2018 in the month of March

Figure 5 and 6 displays the goodness of fit of the selected distribution to the data on the rainfall reported at the Los Angeles Civic Centre from 1943 to 2018 in the month of March. The next data is on the infant mortality rate per 1000 live birth for some selected countries Onyekwere, C.K., Okoro, C.N., Obulezi, O.J., Udofia, E.M., & Anabike, I.C. (2022).

Table 4 Infant Mortality Rate per 1000 live birth for some selected Countries

56	10	22	3	69	6	7	11	4	4	19	13	7	27	12	3	4	11	84	27
25	6	35	14	11	12	6													

In Table 5, we fit the Fuyi distribution including some other distributions to the data on the infant mortality rate per 1000 live birth for some selected countries.

Table 5 Analytical measures of performance and fitness using the Infant Mortality Rate per 1000 live Birth

Distr.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	θ	β	K-S	P-value
Fuyi	-70.78	143.552	143.658	145.242	144.163	0.1185	0.7720	1.9611	-	0.1126	0.6908
Burr XII	-92.86	189.719	190.043	193.097	190.940	0.6770	3.7911	3.0168	0.3077	0.3232	4.71e-4
NEX	-71.34	146.678	147.002	150.056	147.899	0.1672	1.0583	0.0183	2.7195	0.1310	0.4994
EIE	-79.68	163.350	163.675	166.728	164.572	0.4130	2.4340	0.9779	2.2108	0.2210	0.0403
Weibull	-69.56	143.116	143.441	146.494	144.338	0.1183	0.7705	2.5055	3.5164	0.1187	0.6256
Gamma	-73.55	151.275	151.598	154.653	152.497	0.2413	1.4844	3.1673	0.9847	0.1658	0.2214
Lomax	-85.78	175.556	175.881	178.934	176.778	0.2434	1.4964	60676478	19320525	0.30023	1.477e-3
GIE	-85.84	175.690	176.014	179.068	176.911	0.5972	3.3875	2.6374	3.6035	0.2268	0.0327
Lnormal	-79.03	166.040	166.364	169.418	167.261	0.4054	2.3850	1.0074	0.8221	0.2057	0.0679

Based on the results in table 5, the proposed Fuyi distribution fits the infant mortality rate per 1000 live birth for some countries having the highest p-value= 0.6908. However, in parameter estimation it performed better than all competing distributions except the Weibull distribution. The performance measure for the proposed distribution, Fuyi distribution, had the least information criterions among others, except for that of the Weibull distribution where it had its AIC and CAIC values slightly higher and its BIC and HQIC slightly lower than that of the Weibull distribution.

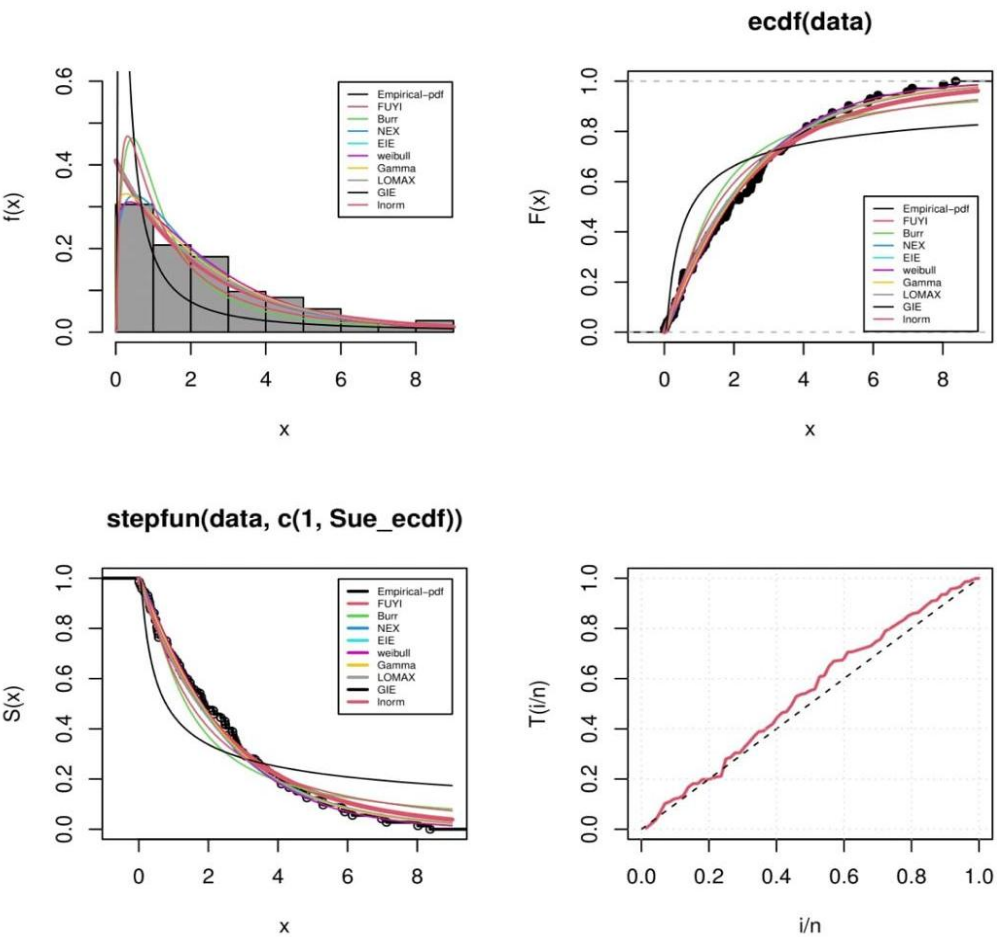


Fig. 7 Density, cdf, survival, and TTT plots for the Infant Mortality Rate Per 1000 live Birth for Some Selected Countries

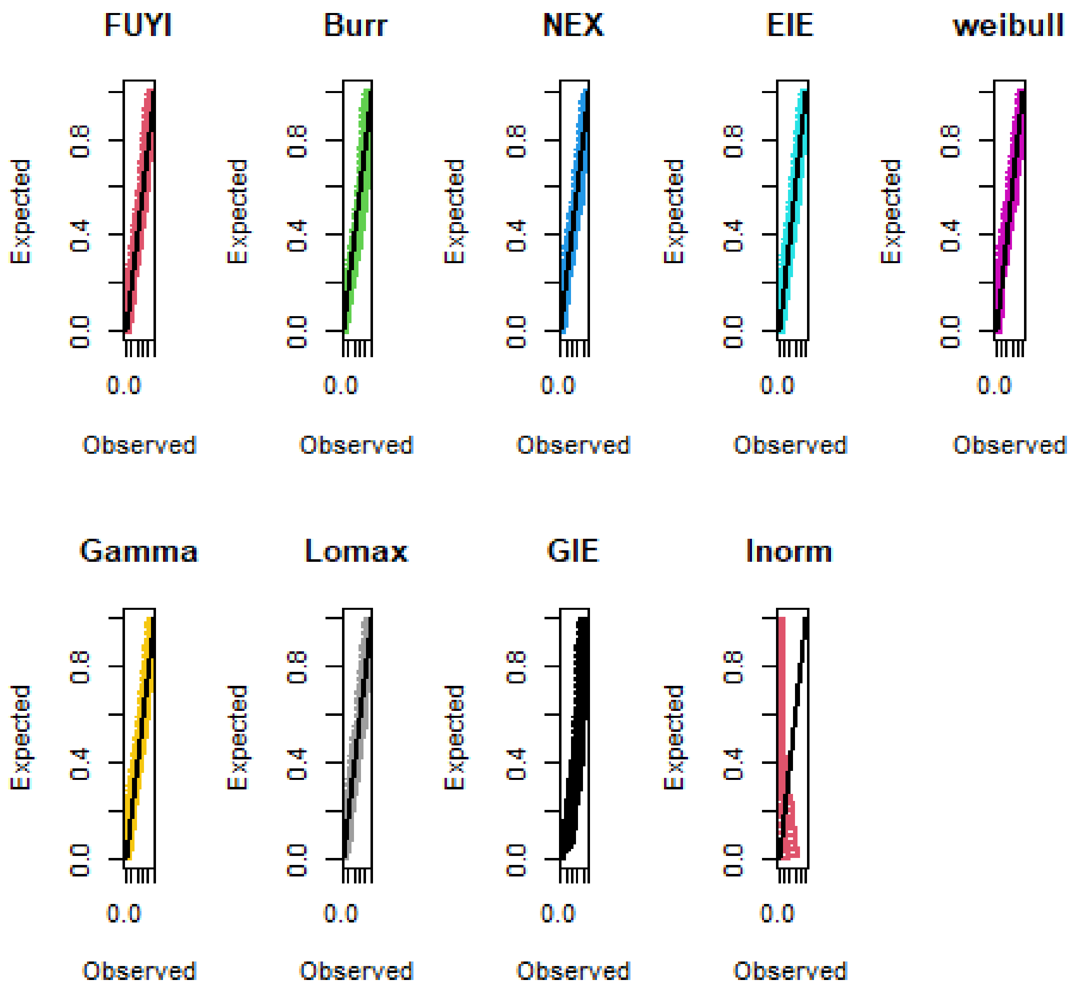


Fig. 8 PP plots for the Infant Mortality Rate Per 1000 live Birth for Some Selected Countries

Figure 7 and 8 displays the goodness of fit of the selected distribution to the data on the Infant Mortality Rate Per 1000 live Birth for Some Selected Countries.

CONCLUSION

In this article, we have proposed a new one parameter distribution which is more flexible in applications using data sets in table 2 and 4. The mathematical properties were derived and the proposed distribution parameter was estimated using the maximum likelihood estimation. Analysis for performance of parameter estimation and goodness of fit was carried by comparing the proposed distribution with eight other existing distributions. On application to the data on rainfall reported at the Los Angeles Civic Centre from 1943 to 2018 in the month of March and Infant Mortality Rate Per 1000 live Birth for some Selected Countries, the Fuyi distribution performed better in parameter estimation and in goodness of fit than the other competing distribution, except for the Weibull distribution, where it only performed better than in goodness of fit test.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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