DOI https://doi.org/10.5281/zenodo.10049652#180

TWIST, 2024, Vol. 19, Issue 2, pp. 471-478

TWIST

Journal homepage: www.twistjournal.net

Extended Runge-Kutta Method (ERKM) Algorithm for the Solution of Optimal Control Problems (OCP)

Akinmuyise, Mathew Folorunsho
Department of Mathematics, University of Education, Ondo, Ondo State, Nigeria

Adebayo, Kayode James*
Department of Mathematics, Ekiti State University, Ado Ekiti, Ekiti State, Nigeria
[**Corresponding author*]

Adisa, Folorunsho Olabisi
Department of Mathematics, University of Education, Ondo, Ondo State, Nigeria

Olaosebikan, Emmanuel Temitayo
Department of Mathematics, Bamidele Olumilua University of Education, Science, and Technology, Ikere Ekiti, Ekiti State, Nigeria

Gbenro, Sunday Oluwaseun
Department of Mathematics, Bamidele Olumilua University of Education, Science, and Technology, Ikere Ekiti, Ekiti State, Nigeria

Dele-Rotimi, Adejoke Olumide
Department of Mathematics, Bamidele Olumilua University of Education, Science, and Technology, Ikere Ekiti, Ekiti State, Nigeria

Obayomi, Abraham Adesoji
Department of Mathematics, Ekiti State University, Ado Ekiti, Ekiti State, Nigeria

Ayinde, Samuel Olukayode
Department of Mathematics, Ekiti State University, Ado Ekiti, Ekiti State, Nigeria

Abstract

This paper discusses the adoption of the Runge-Kutta Method which is classically designed for solving First Order Differential equations to the solution of Optimal Control Problems (OCP) constrained by state differential equations. The Extended Runge-Kutta Method (ERKM) algorithm requires the formulation of the Hamiltonian from the given Optimal Control Problems. This will, in turn, be used to generate the appropriate multi-boundary conditions. The developed boundary conditions will then be embedded at each iteration of the ERKM algorithm to determine the values of the state, the co-state, and the control variables until a satisfactorily prescribed tolerance is reached. The ERKM algorithm was tested on some Lagrange forms of the Optimal Control Problems with successes recorded compared to existing results.

Keywords

Runge-Kutta Method, Extended Runge-Kutta Method, Optimal Control Problems, Hamiltonian, Control variable

INTRODUCTION

In recent years, numerous methods have been developed to solve constrained Optimal Control Problems (OCP) that emanated from Engineering, Physical Sciences, and other fields of human endeavors. The most popular and common method is the calculus of variation method [20] and [18] in which the first optimality conditions which the first optimality condition(s) is/are expected to be derived. The condition(s) resulted in a two-point or multi-point boundary with a special structure because it arises from taking the derivative of the Hamiltonian, $\mathcal{H}(x(t), u(t), \lambda(t), t)$ that is of the form:

$$\frac{\partial \mathcal{H}(x(t), u(t), \lambda(t), t)}{\partial \lambda(t)} = \dot{x}(t) \quad (1)$$

$$\frac{\partial \mathcal{H}(x(t), u(t), \lambda(t), t)}{\partial \lambda(t)} = \dot{\lambda}(t) \quad (2)$$

Though the method is good for computation, it becomes more cumbersome when the problems to be solved are complex.

Another method this research is concerned with is the operator-based method as described in [7]. This requires the transformation of the constrained problem into an unconstrained one using the penalty function method as observed in [4], [1], and [5], or the Multiplier method as seen in [7], [17], [22], and [27]. The result obtained from this conversion will be expanded and written in a bi-linear form with the notion that $a_1 = a_2$, $b_1 = b_2$, etc. The applications of a series of theorem that include the Laplace transformation and Convolution integral theorem resulted in the development of the developed operator which replaces the Hessian matrices in the multivariate unconstraint minimization method in Optimization theory. This method is good but the process involved in constructing the control operator could be more relaxed and convenient.

Contrary to the two methods described above, the ERKM algorithm combines some steps involved in the indirect method with the Runge-Kutta Algorithm to develop a preferred and less stressful method of solution that is believed to give a better approximation.

REVIEW OF RELATED LITERATURE

Due to the algorithmic framework of this Research method, the relationship of the ERKM algorithm with the three major methods to be reviewed are:

The Indirect Method, the Operator-Based Method using a multiplier as a method of transformation, and the Runge-Kutta Method.

The Indirect Method of Solving Optimal Control Problems

Let us consider a plant whose system is described by the first-order differential equation as:

$$\dot{x}(t) = f(x(t), u(t), t) \quad (3)$$

with a Bolza form performance measure by [10] and [24] as

$$J(.) = s(x(t))|_{(t=t_f)} + \int_{t_0}^{t_f} v(x(t), u(t), t)dt \quad (4)$$

while transforming (3) and (4) to unconstrained via the Lagrange Multiplier Method according to [16] and [22] gives rise to

$$J(.) = \int_{t_0}^{t_f} v(x(t), u(t), t)dt + \int_{t_0}^{t_f} \left[\left(\frac{\partial s(.)}{\partial t}\right)^T \dot{x}(t) + \frac{\partial s(.)}{\partial t}\right] dt$$
$$+ \int_{t_0}^{t_f} \lambda^T(t)[f(x(t), u(t), t) - \dot{x}(t)]dt \quad (5)$$

According to (1), the Lagrange function form of (5) is represented as

$$\int_{t_0}^{t_f} L(x(t), u(t), \lambda(t),\ t)\, dt \quad (6)$$

where (6) can be defined as

$$\int_{t_0}^{t_f} L(x(t), u(t), \lambda(t),\ t)\, dt = v(x(t), u(t), \lambda(t),\ t) + \lambda^T(t)\left[f(x(t), u(t), t) + \left[\left(\frac{\partial s(x(t),t)}{\partial t}\right)^T \dot{x}(t) + \frac{\partial s(x(t),t)}{\partial t}\right] - \lambda^T(t)\dot{x}(t)\right]$$

this can be put in the Hamiltonian form as

$$\mathcal{H}(x(t), u(t), \lambda(t), t) + \left[\left(\frac{\partial s(x(t),t)}{\partial t}\right)^T \dot{x}(t) + \frac{\partial s(x(t),t)}{\partial t}\right] - \lambda^T(t)\dot{x}(t)$$

where $\mathcal{H}$ is the Hamiltonian Function according to [21] and [23]. Then, if the objective function (5) is perturbed to give

$$J_a(.) = \int_{t_0}^{t_f+\delta t_f} v(x^*(t) + \delta x(t), u^*(t) + \delta u(t), t)dt + \int_{t_0}^{t_f+\delta t_f} \left[\left(\frac{\partial s(.)}{\partial t}\right)^{T*} (\dot{x}^*(t) + \delta\dot{x}(t)) + \left(\frac{\partial s(.)}{\partial t}\right)^*\right] dt$$
$$+ \int_{t_0}^{t_f+\delta t_f} \lambda^T(t)[f(x^*(t) + \delta x(t), u^*(t) + \delta u(t), t) - (\dot{x}^*(t) + \delta\dot{x}(t))]dt$$
$$= \int_{t_0}^{t_f+\delta t_f} L_p(.)dt = \int_{t_0}^{t_f} L_p(.)dt + \int_{t_f}^{t_f+\delta t_f} L_p(.)dt$$
$$\equiv \int_{t_0}^{t_f} L_p(.)dt + \int_{t_0}^{t_f} L_p(.)|_{t=t_f}\delta t_f dt \quad (7)$$

where $L_p(.)$ is the perturbed model of the Lagrange multiplier [22]. The variation of the functional value can be expressed as:

$$\Delta J_a = J_a(.) - J(.) = \int_{t_f}^{t_f+\delta t_f} L_p(.)dt - \int_{t_0}^{t_f} L_p(.)dt \equiv \int_{t_0}^{t_f} L_p(.)dt + L(.)|_{t=t_f}\delta t_f - \int_{t_0}^{t_f} L(.)dt$$

Therefore,

$$\int_{t_0}^{t_f}(L_p(.)dt + L(.))dt =$$
$$\int_{t_0}^{t_f}(L_p(x^*(t) + \delta x(t), u^*(t), t) + \delta u(t), t) - L(x^*(t), u^*(t), \lambda^*(t), t) + \left(x^*(t), u^*(t), \lambda^*(t), t)|_{t=t_f}\delta t_f\right) \quad (8)$$

Applying the Taylor series expansion to (8) and integrating by part as seen in [25] and [20] and as put to use in [19] taking into consideration the first variation of the functional in the (8) gives rise to

$$\delta J = \int_{t_0}^{t_f}\left[\frac{\partial L(.)}{\partial t} - \frac{d}{dt}\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{T*}\delta x(t)\right]dt + \int_{t_0}^{t_f}\left[\left(\frac{\partial L(.)}{\partial u(t)}\right)^{T}\delta u(t) + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{T*}\delta x(t)\right]dt + L(.)^*|_{t=t_f}\delta t_f \quad (9)$$

Lemma 1: Let $g(t)$ and $x(t)$ be continuous and integrable over a close interval t_0 and t_f then $\int_{t_0}^{t_f} g(t)\delta x(t)dt = 0$ at every point over the integral $[t_0,\ t_f]$ as appeared in [19] and referenced by [23].

From (9) using Lemma 1, one obtains

$$\frac{\partial L(.)}{\partial t} - \frac{d}{dt}\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{*} = 0 \quad (10)$$

and

$$\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{*} = 0 \quad (11)$$

Finally,

$$\delta J \approx L(.)^*|_{t=t_f}\delta t_f + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{T*}\delta x(t)|_{t=t_f} \quad (12)$$

$$\delta x_f = \delta x_{t_f} + \dot{x}(t)|_{t=t_f}\delta x_f = \delta x_{t_f} + x^*(t) + \delta \dot{x}(t)|_{t=t_f})\delta x_f \approx \delta x_{t_f} + (\dot{x}(t_f)^*)\delta t_f \quad (13)$$

substituting (13) into (12) gives rise to

$$\delta J = L(.)^*|_{t=t_f}\delta t_f + \left[\left(\frac{\partial L(.)}{\partial u(t)}\right)^{*}\Big|_{t=t_f}\right](\delta u(t) - \dot{x}(t))\delta t_f \quad (14)$$

Simplifying (14) gives rise to

$$\delta J \approx L(.) - \left[\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\dot{x}(t)\right)^{*}\Big|_{t=t_f}\delta t_f\right] + \left[\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{*}\Big|_{t=t_f}\delta t_f\right]$$

If $\delta J = 0$ in (14) which is regarded as a necessary condition, then

$$\delta J \approx L(.) - \left[\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\dot{x}(t)\right)^{*}\Big|_{t=t_f}\delta t_f\right] + \left[\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)^{*}\Big|_{t=t_f}\delta t_f\right] = 0 \quad (15)$$

where

$$L(.) = \mathcal{H}(x(t), u(t), \lambda(t), t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t) \quad (16)$$

Using (16) in (10), (11), and (15), from (10) gives

$$\frac{\partial}{\partial x(t)}\left[\mathcal{H}(x(t), u(t), \lambda(t), t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t)\right]$$
$$-\frac{d}{dx}\left[\frac{\partial[\mathcal{H}(x(t),u(t),\lambda(t),t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t)]}{\partial x}\right] \quad (17)$$

Using the method akin to [14], one obtains

$$\frac{d}{dt}f(.) = \frac{\partial f(.)}{\partial x(t)}\dot{x}(t) + \frac{\partial f(.)}{\partial y(t)}\dot{y}(t) + \frac{\partial f(.)}{\partial z(t)}\dot{z}(t) \quad (18)$$

Applying (18) in (17) gives rise to

$$\frac{\partial}{\partial x(t)}\left[\mathcal{H}(.) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t)\right] - \frac{d}{dt}\left[\frac{\partial s(.)}{\partial x(t)} - \lambda^T(t)\right]$$

where $\frac{\partial s(.)}{\partial \dot{x}(t)}$, and $\frac{\partial s(.)}{\partial t}$ can combined.

From (18), the following is established

$$\left(\frac{\partial L(.)}{\partial x(t)}\right)^{*} = -\dot{\lambda}(t) \quad (19)$$

It should be noted that (19) is called the co-state equation free of $\dot{x}(t)$. Also, from (11)

$$\left(\frac{\partial L(.)}{\partial u(t)}\right)^{*} = 0 \Rightarrow \left(\frac{\partial L(.)}{\partial u(t)}\right)^{*} = \left(\frac{\partial \mathcal{H}(.)}{\partial x(t)}\right)^{*} = 0 \quad (20)$$

where $L(.)$ remains as defined in (16).

A similar version of (20) is given as

$$\left(\frac{\partial \mathcal{H}(.)}{\partial \lambda(t)}\right)^{*} = -\dot{x}(t). \quad (21)$$

If the system is expressed in state space form (21) becomes the state equation.

Finally, the boundary condition (15) can be written in Hamiltonian form similar to [21] as

$$\left[\mathcal{H}(.) + \left(\frac{\partial s(.)}{\partial x(t)}\right)^{T} - \lambda^T(t)\dot{x}(t) - \left[\frac{\partial s(.)}{\partial u(t)} - \lambda(t)\right]\dot{x}(t)\right]^{*}\Big|_{t=t_f}\delta t_f$$
$$+\left[\frac{\partial s(.)}{\partial u(t)} - \lambda(t)\right]^{*}\Big|_{t=t_f} = 0 \quad (22)$$

Therefore,

$$\left[\mathcal{H}(.) + \frac{\partial s(.)}{\partial x(t)}\right]^{*}\Big|_{t=t_f}\delta t_f + \left[\left[\frac{\partial s(.)}{\partial x(t)} - \lambda(t)\right]\right]^{*}\Big|_{t=t_f}\delta t_f = 0 \quad (23)$$

If (19) through (21) and (23) are solved, the trajectory, $x(t)$, will be gotten, and the control input, $u(t)$, that minimizes the performance measure/index in (4) be the algorithmic steps given the constraints (3) and (4) then,
Step 1: Form the Hamiltonian from the optimal control problem.
Step 2: Compute the co-state in (19).
Step 3: Determine the control input using (20).
Step 4: Compute the state value in (21).
Step 5: Determine the Trajectory in (23).
Step 6: Determine the solution to problems (3) and (4) using the values in steps 2 through 5.
Step 7: Test for convergence. If the convergence condition is satisfied then stop, else, repeat steps 2 through 6.

Operator Based Method

Over the years, several researchers have worked on the construction of different operators. These operators are introduced to either the Conjugate Gradient Method (CGM) as in [2] and [3] or the Quasi-Newton Methods as in [6] of solving Optimal Control Problems. The development of the operator requires a broad knowledge of transforming a constrained optimal control problem into an unconstrained problem. Let us consider the general optimal control problem by [18] and [3] of the form:

$$Minimize\ J = \int_{t_0}^{t_f}[x^T(t)px(t) + u^T(t)qu(t)]\,dt \qquad (24)$$

$$\text{Subject to } x'(t) = cx(t) + du(t),\ t_0 \le t \le t_f \qquad (25)$$

$$0 \le t_0 \le t_f \qquad (26)$$

To support the operator construction, the following are necessary.

Definition 1: Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces over $\mathcal{R}$. A bilinear form of functional Q on $\mathcal{H}_1 \times \mathcal{H}_2$ is a mapping $Q: \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{R}$ such that for all $x, x_1, x_2 \in \mathcal{H}_1$, for all $y, y_1, y_2 \in \mathcal{H}_2$ and $\alpha, \beta \in \mathcal{R}$ then by [11] the following hold:

1. $Q(x_1 + x_2, y) = Q(x_1, y) + Q(x_2, y)$
2. $Q(x, y_1 + y_2) = Q(x, y_1) + Q(x, y_2)$
3. $Q(\alpha x, y) = \alpha Q(x, y)$
4. $Q(x, \beta y) = \beta Q(x, y)$
5. $Q(x, y = Q(y, x)$ for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ on $\mathcal{H}$, Q then Q is called a bilinear, Hermitian (self-adjoin) form on $\mathcal{H}$

Theorem 1. let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\mathcal{H}_1 \times \mathcal{H}_2 \in \mathcal{R}$ be a bounded bilinear form then, for $x \in \mathcal{H}_1$ and $x \in \mathcal{H}_2$, Q has a representation

$$\mathrm{Q(x, y) = Q < sx, y >} \mathcal{H}_2 \qquad (27)$$

where $S: \mathcal{H}_1 \to \mathcal{H}_2$ is a uniquely determined bounded operator. If $T: \mathcal{H} \to \mathcal{H}$ is a continuous linear operator on the Hilbert space $\mathcal{H}$ and for fixed $y \in \mathcal{H}$ and $x \in \mathcal{H}$ then, $f_y(x) = \langle Tx, y\rangle_{\mathcal{H}}$ where f_y defines a continuous linear functional on $\mathcal{H}$.

The expansion and bilinear form of (27) according to Definition 2.1 gives rise to

$$\langle z, Az\rangle_{\widehat{\mathcal{H}}} = \int_{t_0}^{t_f}[x_1(t)p(t)x_2(t) + u(t)q(t)u_2(t) + \alpha x_1'(t)x_2'(t) - \alpha c x_1'(t)x_2(t) - \alpha d x_1'(t)u_2(t) - \alpha c x_1(t)x_2'(t) + \alpha c^2 x_1(t)x_2(t) + \alpha c d x_1(t)u_2(t) - \alpha d u_1(t)x_2'(t) + \alpha c d u_1(t)x_2(t) + \alpha d^2 u_1(t)u_2(t) + \lambda_1(t)x_2'(t) + x_1'(t)\lambda_2(t) - c\lambda_1(t)x_2(t) - cx_1(t)\lambda_2(t) - d\lambda_1(t)u_2(t) - du_1(t)\lambda_2(t)]dt \qquad (28)$$

Suppose $\lambda_2(t) = u_2(t) = 0$ in (28). This reduces (28) to

$$\langle z, Az\rangle_{\widehat{\mathcal{H}}} = \int_{t_0}^{t_f}[x_1(t)p(t)x_2(t) + \alpha x_1'(t)x_2'(t) - \alpha c x_1'(t)x_2(t) - \alpha c x_1(t)x_2'(t) + \alpha c^2 x_1(t)x_2(t) - \alpha d u_1(t)x_2'(t) + \alpha c d u_1(t)x_2(t) + \lambda_1(t)x_2'(t) - c\lambda_1(t)x_2(t)]dt \qquad (29)$$

$$\langle z, Az\rangle_{\widehat{\mathcal{H}}} = \int_{t_0}^{t_f}[x_1(t)p(t)x_2(t) - \alpha c x_1'(t)x_2'(t) + \alpha c^2 x_1(t)x_2(t) + \alpha x_1'(t)x_2'(t) - \alpha c x_2(t)u_1(t) - \alpha d x_2'(t) + \alpha c d u_1(t)x_2(t) + \lambda_1(t)x_2'(t) - c\lambda_1(t)x_2(t)]dt \qquad (30)$$

From (28), one of the entries of the operator is given as

$$u_1(t)A_{21}(t) = u_1(t)[-\alpha d x_2'(t) + \alpha c d u_1(t)x_2(t)] \qquad (31)$$

Dividing (31) through by $u_1(t)$ implies that

$$A_{21}(t) = -\alpha d x_2'(t) + \alpha c d u_1(t)x_2(t) \qquad (32)$$

Consequently,

$$A_{31}(t) = x_2'(t) - cx_2(t) \qquad (33)$$

To determine the first component of the operator as in [2] and [3], one assigns

$$\gamma(t) = p(t)x_2(t) - \alpha c x_2'(t) + \alpha c^2 x_2(t) \qquad (34)$$

and

$$\beta(t) = \alpha c x_2'(t) - \alpha c x_2(t) \qquad (35)$$

The following function

$$\left.\begin{matrix}\gamma(t) - A_{11}(t)\\ \beta(t) - A_{11}'(t)\end{matrix}\right\} \qquad (36)$$

are continuous function on the closed interval $[t_0, t_f]$ also, for $x_1(.) \in A[t_0, t_f]$ such that $x_1(t_f) = 0$ then gives rise to

$\int_{t_0}^{t_f} x_1[\gamma(t) - A_{11}(t)] + x_1'(t)[\beta(t) - A_{11}'(t)]dt$ (37)

So that $\beta(t) - A_{11}'(t)$ is differentiable on the interval $[t_0, t_f]$ with

$\frac{d}{dx}[\beta(t) - A_{11}'(t)] = \gamma(t) - A_{11}(t)$ (38)

On differentiating and evaluating (38), one obtains

$A_{11}'(t) - A_{11}(t) = \beta'(t) - \gamma(t)$ (39)

Setting the LHS of (39) gives rise to

$A_{11}'(t) - A_{11}(t) = v(t)$ for $t_0 \leq t \leq t_f$ (40)

Transforming (40) via Laplace as reported by [2] leads to

$A_{11}(t) = \frac{v(s)}{s^2-1} + \frac{n_0(s)}{s^2-1} + \frac{m_0}{s^2-1}$ (41)

Equivalently, on solving (41) can then be written as

$\int_{t_0}^{t} v(\tau)Sinh(t-\tau)d\tau + n_0 Cosh\, t + m_0 Sinh\, t$ (42)

One needs to determine the prescribed constants in a manner akin to the method used in [17] and [5] as

$n_0 = px(0) - \alpha cx'(0) + \alpha c^2 x(0)$ (43)

$m_0 = \frac{1}{Sinh\, t_f}\left[[px(t_f) - \alpha cx'(t_f) + \alpha c^2 x(t_f) - \int_{t_0}^{t} v(s)Sinh(t_f - s)ds] - [px(0) - \alpha cx'(0) + \alpha c^2 x(0)]Cosh\, t_f\right]$

(44)

On substituting (43) and (44) in (41), it gives

$$A_{11}(t) = -[\alpha x'(0) - \alpha x(0)]Sinh\, t + \int_{t_0}^{t}[\alpha x'(s) - \alpha x(s)]Cosh\,(t-s)ds - \int_{t_0}^{t}[px(s) - \alpha cx'(s) + \alpha xc^2(s)]Sinh\,(t-s)ds + [px(0) - \alpha cx'(0) + \alpha c^2 x(0)]Cosh\, t +$$

$\frac{Sinh\, t}{Sinh\, t_f}\left[[px(t_f) - \alpha cx'(t_f) + \alpha c^2 x(t_f)] + [\alpha x'(0) + \alpha cx(0)]Sinh\, t_f - \int_{t_0}^{t}[\alpha x'(s) + \alpha cx(s)]\, Cosh\,(t_f - s)ds + \int_{t_0}^{t}[px(s) - \alpha cx'(s) + \alpha c^2 x(s)]\, Sinh\,(t_f - s)ds - [px(0) - \alpha cx'(0) + \alpha c^2 x(0)]Cosh\, t_f\right]$ (45)

Repeating the same procedures from (34) to (44) that led to (45) in a similar manner, one obtains the components $A_{12}(t)$, $A_{22}(t)$, and $A_{32}(t)$ as

$A_{22}(t) = qu(t) + \alpha d^2 u(t)$ (46)

$A_{32}(t) = -u(t)$ (47)

and

$A_{12}(t) = \alpha du(0)Sinh\, t_f + \int_{t_0}^{t}[-\alpha du(s)Cosh\,(t_f - s)]ds - \int_{t_0}^{t}[\alpha cdu(s)Sinh\,(t_f - s)]ds + [\alpha cdu(0)Cosh\, t] + \frac{Sinh\, t}{Sinh\, t_f}[\alpha cdu(t_f)Sinh\,(t_f)] + \int_{t_0}^{t}[\alpha du(s)Cosh\,(t_f - s)]ds + \int_{t_0}^{t}[\alpha du(s)Sinh\,(t_f - s)]ds - \alpha cdu(0)Cosh\, t_f$ (48)

Finally,

$A_{23}(t) = -d\lambda(t)$ (49)

$A_{33}(t) = 0$ (50)

and $A_{13}(t)$ can be determined thus

$A_{13}(t) =$

$[-\lambda(0)Sinh\, t_f] + \int_{t_0}^{t}[\lambda(s)Cosh\,(t_f - s)]ds - \int_{t_0}^{t}[c\lambda(s)Sinh\,(t_f - s)]ds + [c\lambda(0)Cosh\, t_f] + \frac{Sinh\, t}{Sinh\, t_f}[c\lambda(t_f) + \lambda(0)Sinh\, t_f]$ (51)

$-\int_{t_0}^{t}[\lambda(s)Cosh\,(t_f - s)]ds + \int_{t_0}^{t}[c\lambda(s)Sinh\,(t_f - s)]ds$
$+[c\lambda(0)Cosh\, t] - [\lambda(0)Cosh\, t_f]$ (52)

$$(Az)(t) = \begin{pmatrix} A_{11}x(t) & A_{12}u(t) & A_{13}\lambda(t) \\ A_{21}x(t) & A_{22}u(t) & A_{23}\lambda(t) \\ A_{31}x(t) & A_{32}u(t) & A_{33}\lambda(t) \end{pmatrix} \quad (53)$$

Hence, (53) is the operator representing the bilinear form of (30). The operator (53) replaces the Hessian matrix in the ECGM as opined by [21], [2], and [3] also, the Newton's Method as found in [6].

Runge-Kutta Method of Solving First-Order Differential Equations

The Runge-Kutta method for solving first-order differential equations has been widely used in numerical analysis according to [12] and affords a degree of accuracy. It is a step-by-step process where a table of function values for a range of independent variables is accumulated. It considers a general form of $y^0 = f(x,y)$ with the initial conditions $x = x_0$, $y = y_0$, $y' = y_0'$ and computed as follows:

$k_1 = hf(x_0, y_0)$ (54)

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$ (55)

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$ (56)

$k_4 = hf(x_0 + h, y_0 + k_3)$ (57)

and the argument Δy_0 in y-values from x_0 to x_1 given as

$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ (58)

and finally,

$y_1 = y_0 + \Delta y_0.$ (59)

EXTENDED RUNGE-KUTTA METHOD ALGORITHM

The application of the ERKM algorithm to optimal control problems requires an in-depth knowledge of classical optimization, Optimal control study, and Numerical analysis. The ERKM algorithm can be summarized as follows:

Step 1: Convert the constrained optimal control problem to an unconstrained one via the Hamiltonian method.

Step 2: Determine the first-order optimality conditions.

Step 3: Determine the two-point or multipoint boundary conditions.

Step 4: Embed the boundary conditions in the algorithm of the Runge-Kutta method to determine the numerical value of the state and the co-state variables.

Step 5: If $\frac{\partial \mathcal{H}}{\partial u} \leq 0.05$ then stop else, go to step 6.

Step 6: Update u_{i+1} and repeat steps 2 through 5.

The ERKM algorithm fuses the strengths of the operator-based method and Runge-Kutta methods to form a new method for the solution of the OCP.

PROBLEMS, RESULTS, AND DISCUSSIONS

This section presents some optimal control problems to test the efficiency and robustness of the Extended Runge-Kutta Method Algorithm. The results generated shall be discussed via a viz some convergence criteria.

Problem 1: Lagrange Form of Optimal Control Problem without Delay as in [11] and [13].

$Minimize_{(x,u,\lambda)} J = 0.5\int_0^1 [x^T(t)Px(t) + u^T(t)Ru(t)]\,dt$

Subject to $\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_1 + u(t)$

$x_1(0) = 1,\ \ x_2(0) = 0.5,\ \lambda_1(0) = 1,$ and $\lambda_2(0) = 0$

$t_0 \leq t \leq t_f,\ P = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, and $R = 1.$

Problem 2: Lagrange Form of Optimal Control Problem [9] with the weighted matrix as the coefficient

$Minimize_{(x,u,\lambda)} J = \int_0^1 [x^T(t)Px(t) + u^T(t)Ru(t)]\,dt$

Subject to $\dot{x}_1(t) = 2x_2(t)$

$\dot{x}_2(t) = -x_1(t) - 3x_2(t) + u(t)$ and $t_0 \leq t \leq t_f$

$x_1(0) = 2,\ \ x_2(0) = 1,\ \lambda_1(0) = 1,\ \lambda_2(0) = 0,\ R = 1$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Problem 3: Lagrange Form of Optimal Control Problem

$Minimize_{(x,u,\lambda)} J = \int_0^1 [x^T(t)Px(t) + u^T(t)Ru(t)]\,dt$

Subject to $\dot{x}_1(t) = 2x_2(t)$

$\dot{x}_2(t) = -x_1(t) - 3x_2(t) + u(t)$

$x_1(0) = 2,\ \ x_2(0) = 1,\ \lambda_1(0) = 1,\ \lambda_1(0) = 0,\ R = 1$ and $P = 0.$

The solutions to Problems 1, 2, and 3 are presented in Tables 1, 2, and 3 respectively. The results, J^*, are compared with the analytic solutions, $J\ Exact$, of each test problem showing the error differences, $J\ Error$.

Numerical Results of Tested Problems

Table 1 Numerical solution of Problem 1

t	x_1^*	x_2^*	λ_1^*	λ_2^*	u^*	J^*	$J\ Exact$	$J\ Error$
0.2	1.96750	0.10012	1.88670	0.13362	-0.06681	2.06778	2.35203	0.28425
0.4	1.94491	0.00121	1.27682	-0.25904	0.12952	2.34818	2.51499	0.16651
0.6	1.89990	-0.10530	1.10243	-0.39259	0.196295	2.21704	2.70353	0.48649
0.8	1.74840	-0.41960	0.34615	-0.46923	0.234615	2.57666	2.66815	0.09149
0.9	1.73840	-0.40133	0.20413	-0.29994	0.14997	2.48544	2.55687	0.07143
1.0	1.74821	-0.40261	0.001935	-0.000240	0.000220	2.33862	2.32716	0.01146

Table 2 Numerical solution of Problem 2

t	x_1^*	x_2^*	λ_1^*	λ_2^*	u^*	J^*	J Exact	J Error
0.2	2.11987	0.030027	4.00144	2.38901	-2.38901	4.09298	4.19442	0.10144
0.4	2.10189	-0.27197	2.70101	1.29975	-1.29975	4.13927	4.19442	0.05485
0.6	1.97501	-0.24013	1.66890	0.66046	-0.66046	4.08269	4.19442	0.11143
0.8	1.94334	-0.02525	0.80144	0.27038	-0.27038	4.18971	4.19442	0.00441
0.9	1.94776	0.13078	0.41104	0.12711	-0.12711	4.16692	4.19442	0.02720
1.0	1.99899	0.31406	0.00101	0.00012	-0.00012	4.19319	4.19412	0.00093

Table 3 Numerical solution of Problem 3

t	x_1^*	x_2^*	λ_1^*	λ_2^*	u^*	J^*	J Exact	J Error
0.2	1.013795	1.11296	0.97688	0.99867	-0.99867	-0.92256	-0.77017	0.15239
0.4	1.37152	0.96443	1.34522	1.00504	-1.00504	-1.09117	-0.77017	0.32100
0.6	1.44435	0.31035	1.51071	0.73924	-0.73924	-1.14535	-0.77017	0.37518
0.8	1.75658	-0.021961	1.93218	0.39676	-0.39676	-0.89679	-0.77017	0.12662
0.9	1.75238	-0.24013	1.83951	0.20013	-0.20013	-0.83248	-0.77017	-0.06231
1.0	1.71758	-0.41616	1.85082	0.000102	- 0.000102	-0.77041	-0.77017	-0.00024

Discussion on the Performance of ERKM Algorithm

It can be seen from Tables 1, 2, and 3 that the three tested problems have similar characteristics:

(i) value of the state and Performance Measure $x^*(t)$ and $J^*(t)$ changes for different values of t as $t \approx t_f$

(ii) the values of the control $u(t)$ decreased as t was approaching the terminal point i.e. $t \approx t_f$

(iii) as $t = t_f$, then $|J\ Error| \approx 0$.

CONCLUDING REMARKS

From the results in the tables above, one can conclude that the method i.e. the Extended Runge-Kutta method is stable, robust, and reliable as it can handle optimal control problems with multiple constraints. The values of t were fixed in each one-dimensional cycle while the step size was updated with the formula $h_{n+1} = h_n + h_{n-1}$ and stopping conditions $u(t) \leq 0.001$ as $t = t_f$ for the three problems.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for the access granted to their papers.

FUNDING INFORMATION

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

DECLARATION CONFLICT

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

REFERENCES

1. Adebayo, K. J., Aderibigbe, F. M., and Akinmuyise, M. F., (2014), On Application of Modified Gradient to Extended Conjugate Gradient Algorithm for Solving Optimal Control Problems, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-3008, pISSN:2319-7676. Volume 9, Issue 5 (Jan. 2014), pp 30-35.
2. Aderibigbe, F. M. and Adebayo, K. J., (2014), On Construction of a Control Operator Applied to Conjugate Gradient Method in Solving Continuous-Time Linear Regulator Problems with Delay-I, IOSR Journal of Mathematics (IOSR-JM), Vol. 10, pp 01-07.
3. Adebayo, K. J., Aderibigbe, F. M., Ayinde, S. O., Olaosebikan, T. E., Adisa, I. O., Akinmuyise, F. M., Gbenro, S. O., Obayomi, A. A., and Dele-Rotimi, A. O., (2024), On the Introduction of a Constructed Operator to an Extended Conjugate Gradient Method (ECGM) Algorithm, Journal of Xi'an Shiyou University, Natural Science Edition, Vol., 20 (04), pp. 562-570.
4. Aderibigbe, F. M., Adebayo, K. J., (2011), Extended Conjugate Gradient method, Journal of Mathematical Sciences, 23, (1), pp 189-196.
5. Aderibigbe, F. M., (1993), An Extended Conjugate Gradient Algorithm for Control Systems with Delay-I, Advances in Modelling and Analysis, CAMSE Press, 36, (3), pp 51-64.
6. Aderibigbe, F. M., Dele-Rotimi, A. O., and Kayode James Adebayo, (2015), On Application of a New Quasi-Newton Algorithm for Solving Optimal Control Problems. Pure and Applied Mathematics Journal. Vol. 4, No. 2, pp. 52-56. doi: 10.11648/j.pamj.20150402.14
7. Akinmuyise, M. F., (2019), Embedded Multiplier in Newton's method for solving variant Optimal Control problems, Paper presented at the 1st Faculty of Science Annual Conference of Ondo State University of Science and Technology on Sustainable Development Through Scientific Innovation: Problems and Prospect, 16th-18th July 2019.
8. Akinmuyise, M. F., Olagunju, S. O., and Olorunsola, S. A., (2022), Numerical Solution of Optimal Control Problems Using Improved Euler's Method, Abacus Mathematics Science Series, Vol. 49, No 2, pp 338-349.

9. Burghes, D. N. and Graham, A., (1980), Introduction to Control Theory including Optimal Control, Ellis Horwood Ltd, Willey and son, New York.
10. David, G. L., (2003), Optimal Control Theory for Applications, Mechanical Engineering Series, Springer-Verlag, Inc., 175fifth Avenue, New York.
11. Douglas, N. A., (2001), A Concise Introduction to Numerical Analysis. Lecture Notes, Penn State, MATH 597I Numerical Analysis.
12. Frank, L. L., Dragguna, L. V., and Vassilis, L. S., (2012), Optimal Control, 3rd Edition, John Wiley and Sons, Inc., Hololen New Jersey.
13. George M. S., (1996), An Engineering Approach to Optimal Control and Estimation Theory, John Wiley and Sons, Inc.
14. Hale, J., (1977), Theory of Functional Differential Equations, Springer Verlag, New York.
15. Hasdor, L., (1976), Gradient Optimization and Non-linear Control, J. Wiley and Sons, New York.
16. Ibiejugba, M. A., (1982), The construction of Control Operator, An Stiint University, AI. I. Cuza lasi XXVII, 45-48.
17. Ibiejugba, M. A., Otunta, F., Olorunsola, S. A., (1992), The Role of Multiplier in the then Multiplier Method, Journal of the Mathematical Society, vol. 11, N. 2, part 3.
18. Kirk, D. E., (2004), Optimal Control Theory, An Introduction, Prentice-Hall, Inc. Englewood Cliffis New Jersey.
19. Luenberger, D. G., (2003), Optimal Control: Introduction to Dynamic Systems, John Wiley and sons, New York,
20. Niclas, A., Anton, E., and Michael, P., (2013), An Introduction to Continuous Optimization, oasis publication pvt Ltd, India.
21. Oke, M. O., Oyelade, T. A., Adebayo, K. J., and Adenipekun, E. A., (2021), On a Modified Higher Order Form of Conjugate Gradient Method for Solving Some Optimal Control Problem, Global Science Journal, Volume 9, Issue 6, pp 244-251.
22. Olorunsola, S. A., Olaosebikan, T. E., and Adebayo, K. J., (2014), Numerical Experiments with the Lagrange Multiplier and Conjugate Gradient Methods (ILMCGM). American Journal of Applied Mathematics. Vol. 2, No. 6, pp. 221-226. doi: 10.11648/j.ajam.20140206.15
23. Price, K. R., Storn, J. A., and Lampinen, M., (2005), Differential Evolution: A Practical Approach to Global Optimization. Springer, ISBN 978-3-540-20950-8.
24. Ray, G. D., (2014), Optimal Control Theory, Lecture Note, Department of Electrical Engineering, Indian Institute of Technology, Kharagpur.
25. Siertrar, B., (2009), Optimal control theory with Economic Applications, Elsevier Science B. V., Amsterdam-Lausaane, New York.
26. Storn, R., (1996), On the Usage of Differential Evolution for Function Optimization. Biennial Conference of the North American Fuzzy Information Processing Society (NAFIPS), 8: 519523.
27. Tripathy, S. S. and Narendra, K. S., (1972), Constrained Optimization problems using Multiplier methods, Journal of Optimization theory and applications, 9, (1), 59-70.
28. Zwillinger, D., (1997), Handbook of Differential Equations, 3rd edition, Academic Press, U. S. A.