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Topological Deconfinement 3D Noncompact Lattice Abelian-Higgs Model Transitions

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Abstract
We discussed the numerical analysis of the dynamics of symmetry breaking in three spatial dimensions for both abelian and non-abelian Higgs models. The topological excitations in the abelian Higgs model as well as in the other field theoretic models that will be described originate from the non-trivial topology of the manifold of vacuum field configurations. We investigate the topological phase transitions that take place in three-dimensional multicomponent lattice Abelian-Higgs (LAH) models that is minimally coupled to a noncompact Abelian gauge field.

Keywords
Abelian Higgs models, Topological Deconfinement Transitions, Homotopy

INTRODUCTION

The emergence of general relativity and quantum field theory, two of theoretical physics' greatest ground-breaking innovations, occurred in the first decade of the 20th century. Major changes were made to the classical field theories' seemingly full representation of our world in a straightforward Euclidean geometrical setting: The classical field theories had to be quantized, and Riemannian geometry replaced Euclidean geometry. These concepts gave rise to the current theories of gravitation and elementary particles, which describe nature more accurately than any previous theories known to physicists. Both theories have a strikingly great number of good predictions, but they also have a strikingly big number of errors. The framework of quantum field theory is used to describe the standard model of elementary particles. We must first quantize a classical field theory in order to build a quantum field theory. Before calculating the physical features of the theory, we must apply a regularization scheme and demonstrate renormalizability because calculations in the quantized theory are plagued with divergences. All of these steps must be taken carefully, and they are obviously not independent of one another. Additionally, it is not quite clear how to make general relativity and the standard model of basic particles compatible. Given that both theories are expressed in a completely distinct mathematical language, this endeavor is quite challenging to approach.

Many strides have been made toward resolving these issues since the 1970s. It's interesting to note that many of these contributions' essential components have to do with topological structures, making topology today an essential component of theoretical physics. Think about the quantization of a gauge field theory, for instance. One chooses a specific gauge to quantize such a theory in order to eliminate extra degrees of freedom. During this process, the symmetry attribute of gauge invariance is lost.

This is disastrous for the renormalizability proof since the terms that appear in the renormalized theory must be constrained by gauge invariance. This issue is resolved by BRST quantization using ideas from algebraic geometry. More generally, the BRST formalism offers a beautiful framework for handling constrained systems, such as those in string theories or general relativity. After the theory has been quantized, we can ask for the symmetries that the quantum field theory inherits from the classical theory.

Surprisingly, when gauge fields couple to the two fermion chiral components differently, the so-called chiral anomalies, one discovers challenges to the formulation of quantized gauge theories. This conundrum is related to the challenges of regularizing such chiral gauge theories without chiral symmetry violation. In terms of local symmetries, physical theories must be anomaly-free. The standard model, whose electroweak sector is a chiral gauge theory, has its couplings and particle composition constrained by this, making it of essential importance. Up until recently, the discussion of anomalies was only perturbative because exact chiral symmetry could not be realized on the lattice, and one might have been concerned about issues with anomaly cancelations that went beyond perturbation theory [1].

Additionally, this problem prohibited a numerical analysis of pertinent quantum field theories. Domain wall, overlap, and perfect action fermions, or more generally, Ginsparg-Wilson fermions, are new lattice regularization schemes that have recently been found to be consistent with a generalized type of chiral symmetry.

C. F. Gauß explains a deep topological finding that he drew from the investigation of a physical problem in a fragment [2] published in the year 1833. In the magnetic field B produced by a current I flowing over a closed loop C_2, he takes into account the work Wm accomplished by moving a magnetic monopole with a magnetic charge g along a closed channel C_1, in that field. W_m, is given by the Biot-Savart law as follows:

$$W_n = g \oint B(x_1) dx_1 = \frac{4\pi g}{c} I \; lk\{C_1, C_2\} \quad (1)$$

where $lk\{C_1, C_2\} = \frac{1}{4\pi} \oint \oint \frac{dx_1 dx_2 x_{12}}{|x_{12}|^3}$

and $x_{12} = x_2 - x_1$

Gauß realized that W_m does not depend on geometric details of the current carrying loop C_2 nor on those of the closed path C_1.

The Linking Number, $lk\{C_1, C_2\}$, does not vary in value under continuous deformations of these curves. An invariant of topology describes this quantity. It is an integer that represents the (signed) total number of times the loop C_1 crosses every given (oriented) surface in $\mathbb{R}^3$ whose edge is the loop C_2 [3] and [4]. Gauß laments the lack of progress made in topology ("Geometria Situs") since the time of Leibniz, who proposed "another analysis, purely geometric or linear which also defines the position (situs), as algebra defines magnitude" in 1679. Leibniz also considered using this new area of mathematics in physics. But he was unsuccessful in trying to persuade Christiaan Huygens, a physicist, to agree with his beliefs about topology. With the formulation of the Kelvin circulation theorem (1869) and the Helmholtz laws of vortex motion (1858), topological arguments first appeared in physics. To this day, hydrodynamics remains a fruitful area for the development and use of topological approaches in physics.

As a result of the success of the topological arguments, Kelvin looked for a way to describe the atoms that made up matter at the time in terms of vortices and, in doing so, to explain topologically why they were stable. Although the first of many to follow topological attempts to explain the principles of fundamental physics had to fail, P. Tait's taxonomy of knots and links emerged from these efforts [5]. In physics nowadays, topological approaches are frequently used to analyze the properties of systems. The Aharonov-Bohm effect and Berry's phase, as well as the stability of defects in condensed matter systems, quantum liquids, and cosmology, have topological roots.

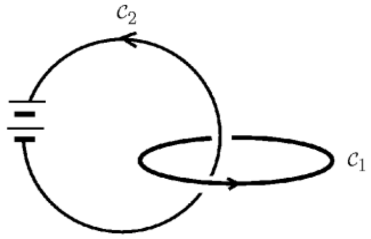

Fig. 1

Topological approaches are insensitive to specifics of the systems under consideration by their very nature. Thus, their use frequently reveals surprising connections between occurrences that at first glance appear to be extremely different. This common ground in the theoretical description applies to less concrete ideas as well as more visible topological phenomena like vortices, which are found on practically all sizes in physics. For example, the topological invariant known as "Helicity" in inviscid fluids, which was discovered in 1969 [6], is strongly related to the topological charge in gauge theories.

Nematic liquid crystal flaws are closely related to flaws in some gauge theories. Topological methods weren't widely used in field theoretic investigations until the formulation of non-abelian gauge theories [7] with their wealth of non-perturbative phenomena. Dirac's work on magnetic monopoles announced the relevance of topology for field theoretic studies in physics in 1931.

LAGRANGIAN ABELIAN HIGGS MODEL

A Nielsen-Olesen vortex [8] in theoretical physics is a point-like entity localized in two spatial dimensions, or, alternatively, a classical field theory solution having the same property. If non-contractible circles are present in the configuration space of scalar fields, then this solution applies. In the configuration space, a circle that surrounds the vortex at infinity may be "wrapped" once on another circle. The Nielsen-Olesen vortex is a structure with this complex topological characteristic, named for Holger Bech Nielsen and Poul Olesen (1973). Formally, the solution is the same as the quantum vortex in superconductor solution. It is of considerable interest to look for analytical solutions to the classical equations of motion of field theories that are relevant to certain areas of physics. The Abelian Higgs model is one of these field theories, and it is significant for particle physics, condensed matter physics, and cosmology [8, 9]. To yet, attempts to solve the complete Euler-Lagrange equations of motion for this model have been unsuccessful. This is due to the fact that these second order partial differential equations are linked and very nonlinear.

As a gauge theory, the abelian Higgs model is used. It also comprises a self-interacting scalar field called the Higgs field, which is only weakly related to the electromagnetic field. It is helpful from a conceptual standpoint to think of this field theory in 2 + 1 dimensional space-time and to then expand it to 3 + 1 dimensions for applications.

The Lagrangian abelian Higgs model is;

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\phi\right)^* \left(D_\mu\phi\right) - V(\phi) \qquad (2)$$

The complex (charged), self-interacting scalar field ϕ is present in this model.

The Higgs entropy;

$$V(\phi) = \frac{1}{4}\lambda(|\phi|^2 - a^2)^2 \qquad (3)$$

where ϕ is the matter field.

Fig. 2 depicts the Higgs potential as a function of the real and hypothetical Higgs field.

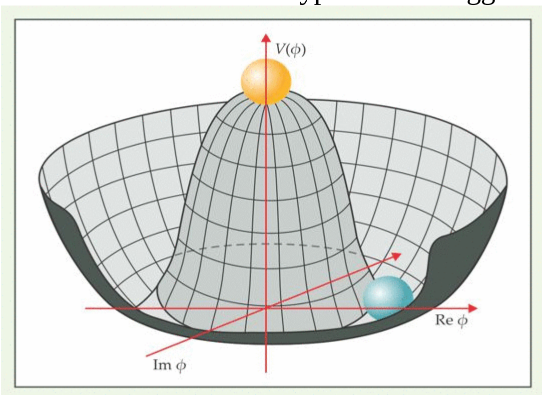

Fig. 2

By design, this Higgs potential is minimal along the complex plane's circle $|\phi| = a$. For stability purposes, it is assumed that the constant, which governs the strength of the Higgs field's self-interaction, is positive λ.

The covariant derivative replaces the partial derivative ∂_μ in the Higgs field's minimal coupling to the radiation field A_μ.

$$D_\mu = \partial_\mu + ieA_\mu \qquad (4)$$

Gauge fields is related to field strengths by;

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{ie}[D_\mu, D_\nu] \qquad (5)$$

MAXWELL HOMOGENEOUS AND INHOMOGENEOUS EQUATIONS

The Lorentz force law and a set of linked partial differential equations known as Maxwell's equations [10], sometimes known as Maxwell-Heaviside equations, serve as the theoretical cornerstones of classical electromagnetism, classical optics, and electric circuits. For electrical, optical, and radio technologies including power generation, electric motors, wireless communication, lenses, radar, etc., the equations offer a mathematical representation. They explain how charges, currents, and changes in the fields produce electric and magnetic fields.

The homogeneous Maxwell equations ensure that the field strength can be stated in terms of the gauge fields even if they are not dynamical equations of motion. The covariant derivative's Jacobi identity leads to the homogeneous equations.

The principle of least action is used to derive the (inhomogeneous) Maxwell equations,

$$\delta S = \delta \int d^4x\mathcal{L} = 0 \qquad (6)$$

using variation of S w.r.t the gauge fields.

$$\text{Now, } \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} = -F^{\mu\nu} \qquad (7)$$

$$\text{And } \frac{\delta\mathcal{L}}{\delta A_\nu} = -j^\nu \qquad (8)$$

where $j_\nu = ie(\phi^*\partial_\nu\phi^*) - 2e^2\phi^*\phi A_\upsilon$).

Repetitive variables are present in gauge theories. The presence of local symmetry transformations—also known as "gauge transformations"—causes this redundancy to become apparent

$$U(x) = e^{ie\alpha(x)} \qquad (9)$$

(9) shifts the value of the gauge field and spins the matter field's phase in a way that depends on place and time as the following

$$\phi \longrightarrow \phi^U = U(x)\phi(x) \quad (10)$$

$$A_\mu \longrightarrow {A_\mu}^U = A_\mu + U(x)\frac{1}{ie}\partial_\mu U(x) \quad (11)$$

Now, D_μ the covariant derivative was defined $D_\mu\phi(x) \longrightarrow U(x)D_\mu\phi(x)$ transforms like ϕ.

This transformation condition ensures that $\mathcal{L}$ and the equations of motion are invariant along with the invariance of $F_{\mu\nu}$.

Quantum Field Theory (QFT) allows for the approximate study of classical statistical systems. The accuracy of the approximation increases with the correlation length in lattice units. Changing the statistical model's temperature equates to deforming the QFT by a specific operator, often one that takes into account all symmetries.

The easiest way to define a renormalization group theory transformation is using the one-dimensional Ising model. The key point is that the renormalization theory can be applied to Ising models in higher dimensions easily, unlike the previously discussed transfer matrix method or other procedures, which makes it instructive to recover the same conclusion from the perspective of renormalization group theory. We have already seen that the one-dimensional Ising model does not present a phase transition using the transfer matrix method.

TOPOLOGICAL DECONFINEMENT TRANSITIONS OF 3-DIMENSIONAL NONCOMPACT LATTICE ABELIAN-HIGGS (LAH) MODELS

Effective three-dimensional (3D) scalar Abelian gauge models, in which scalar fields are coupled with an Abelian gauge field, are used to describe many emergent collective phenomena in condensed matter physics [11, 12].

In order to determine the potential universality classes of the continuous transitions that take place in generic scalar gauge systems, a number of lattice scalar gauge models have been taken into account, utilizing both compact and non-compact gauge variables. They give examples of topological transitions caused by long-range scalar fluctuations and nonlocal topological gauge modes, or by extended charged excitations with no local order parameter.

The fundamental variables in the LAH model are N component complex vectors $(u_x.\overline{u_x})$=1 with the non-compact gauge variables

$A_{x,\mu} \in \mathbb{R}$ where $\mu = 1,2,3$

In the abelian Higgs model as well as in the other field theoretic models to be explored later, the topological excitations originate from the non-trivial topology of the manifold of vacuum field configurations. We proceed in the same manner as when discussing ground state configurations and take into account static fields [13], but we also allow for energy densities that do not completely disappear. Finite energy can only be produced if asymptotically $|x| \longrightarrow \infty$, which follows directly from the expression [14] for the energy density. According to

$$D_{\phi(x)} = (\nabla - ieA(x))\phi(x) \longrightarrow 0 \quad (12)$$

The gauge field is asymptotically defined by the scalar field phase

$$A(x) = \frac{1}{ie}\nabla\, ln\phi(x) = \frac{1}{e}\nabla\theta(x) \quad (13)$$

The vector potential is by design asymptotically a pure gauge [15] and $A(x)$ is not connected with any magnetic field strength.

The asymptotic gauge field's structure (13) leads to the conclusion that the magnetic flux of field configurations with finite energy is quantized. When Stokes' theorem is applied to a surface Λ that is enclosed by the asymptotic curve C, the result is

$$\phi_K^n = \int K\, d^2x = \oint A\, ds = \frac{1}{e}\oint \nabla\theta(x)ds = 2n\frac{\pi}{e} \quad (14)$$

Since ϕ_K^n is an integer multiple of the fundamental unit of magnetic flux, it is a conserved quantity that does not vary as a function of time. This preserved quantity's appearance has topological roots rather than an underlying symmetry as the cause of its manifestation. Since it cannot be altered by a continuous deformation of the asymptotic curve, ϕ_K^n is also regarded as a topological invariant. We suppose that the asymptotic curve C is a circle to highlight the topological significance of this finding.

$|\phi| = a$ on this circle. As a result, the scalar field $\phi(x)$ provides a mapping from the asymptotic circle C to the Higgs potential's circle of zeros ($V(a) = 0$).

HOMOTOPY

The vacuum degeneracy, from a physics perspective, is the fundamental characteristic of the abelian Higgs model that ultimately leads to the quantization of the magnetic flux and the development of topological excitations. Formally speaking, one thinks about fields like the Higgs field as offering a mapping from the configuration space asymptotic circle to the space of zeros of the Higgs potential. In this sense, the quantization results from the association of this mapping's topological invariants with integer values.

While these characteristics are almost obvious in the abelian Higgs model, the structure of the spaces to be mapped in the upcoming applications is more complex. The space of zeroes of the Higgs potential, for example, will be a subset of a non-abelian group in the non-abelian Higgs model. More sophisticated mathematical techniques have proven useful for performing the analysis in such circumstances. The idea of homotopy will be crucial in our discussion and for subsequent applications.

Definition 1

(i) If X and Y are smooth manifolds and $f: X \to Y$ is a smooth function, then the homotopy (from Ancient Greek: ὁμός homós) [16] or deformation $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$, $f_t(x) = F(x,t)$ is homotopic to the initial function $f_0 = f$.
The homotopy is the function of the whole cylinder $X \times I$.
The set of smooth functions $X \to Y$ can be divided into equivalence classes (homotopy classes) because the relation of homotopy between functions is an equivalence relation.

(ii) Two functions f and g are homotopic (denoted by $f \sim g$) if they continuously can be deformed into each other.

(iii) Let f be the identity function defined on $\mathbb{R}^n$ and g be the constant function, if f and g are homotopic with the homotopy $F(x,t) = (1-t)x + tx_0$, then they are contractible.

(iv) The two spaces X and Y are homotopically equivalent if $\exists\ f: X \to Y$ is a continuous function and $g: X \to Y$ is the inverse function such that $g \circ f = I_X$ and $f \circ g = I_Y$

Spaces that can be mapped continuously and bijectively onto one another are topologically similar (homeomorphic), and as a result, they share the same connectivity characteristics and are homotopically equal. The opposite is untrue.

In physics, the parameter t is frequently identified as time. Homotopies include classical fields that change with time. Continuous changes in fields are typically associated with limitless energies or densities of energy. An example of a spin wave providing a homotopy of the "spin system" is one that connects an initial configuration $F(x,0)$ with an eventual configuration $F(x,1)$.

The different sectors (equivalence classes) of field configurations are categorized by homotopy theory. Fields in a particular sector can converge into one another as a result of time. Whether the spin configuration can change over time from the ground state configuration may be of interest.

The connectedness features of spaces that are connectedness properties of loops in these spaces are characterized by the fundamental group. The fundamental concept is to allow loops compress to a point in order to find flaws, such as a hole in the plane. Such approaches will encounter topological obstacles because of certain flaws.

Here, arcwise (or path) connected spaces—that is, spaces where any two points can be connected by a path—are taken into consideration.

Definition 2

(i) A loop is a closed path through α_0 in M given by $f: [0,1] \to M$ such that $f(0) = f(1) = \alpha_0$

(ii) Two loops α and β are homotopic (denoted by $f \sim g$) if $\exists h: [0,1] \times [0,1] \to M$ such that
$\alpha(x,0) = f(x), \quad 0 \le x \le 1$
$h(x,1) = \beta(x),$
$h(0,y) = h(1,y) = \alpha_0, \quad 0 \le y \le 1.$

The relevant degrees of freedom are typically characterized by fields that take values in topological groups, such as the Higgs field in either the abelian or non-abelian Higgs model or link variables and Wilson loops in gauge theories. The order parameter in superfluid in the "A-phase," in which the pairing of the Helium atoms happens in p-states with the spins linked to 1, is a key example in condensed matter physics.

CONCLUSION

The topological characteristics of the space where the loops are defined are characterized by the fundamental group by showing how loops behave under continuous deformations. Only a specific class of non-trivial topological properties may be found with this method. The concept of homotopy groups must be expanded to higher dimensions because, as we have already seen, loops in dimensions more than two are unable to identify a point defect. A 2-sphere can surround a pointlike defect in R3 even though a circle cannot.

By appropriately specifying higher dimensional analogs of the (one dimensional) loops, the higher homotopy groups are created.

REFERENCES

1. E. Bick, F. D. Steffen, (2005). Topology and Geometry in Physics. 659 (Springer, Berlin Heidelberg). DOI 10.1007/b100632
2. M.E.Peskin , D.V. Schroeder, (1995). An Introduction to Quantum Field Theory. Westview Press, Boulder.
3. S. Weinberg. The Quantum Theory Of Fields, I, II, and III, (Cambridge University Press, Cambridge 1995, 1996, and 2000)
4. J. Zinn-Justin, (2002). Quantum Field Theory and Critical Phenomena, 4th ed. (Carendon Press, Oxford).
5. R. Wald, (1984). General Relativity (The University of Chicago Press, Chicago)
   R. A. Bertlmann, (1996). Anomalies in Quantum Field Theory (Oxford University Press, Oxford)
6. S. Coleman, (1985). Aspects of Symmetry (Cambridge University Press, Cambridge)
7. H. B. Nielsen, P. Olesen, (1973). Vortex-line models for dual strings. Nuclear Physics B. 61: 45–61. Bibcode:NuPhB.61.45N. DOI:10.1016/0550-3213(73)90350-7
8. W. Edward, (1985). Superconducting strings, Nucl. Phys. B 249, Issue 4, (1985) 557-592.
9. R. A. Serway, J. W. Jewett. (2012). Principles of physics: a calculus-based text, Cengage Learning, 5th ed. p. 809.

10. P. W. Anderson, (1984). Basic Notions of Condensed Matter Physics, (The Benjamin/Cummings Publishing Company, Menlo Park, California).
11. X.-G. Wen, (2004). Quantum field theory of many-body systems: from the origin of sound to an origin of light and electrons, (Oxford University Press).
12. M. Nakahara, (2003). Geometry, Topology and Physics, 2nd ed. (IOP Publishing, Bristol)
13. B. F. Schutz, (1980). Geometrical Methods of Mathematical Physics (Cambridge University
14. Press, Cambridge)
15. H. Forkel, (2023). A Primer on Instantons in QCD. arXiv:hep-ph/0009136
16. Hatcher, (2002). Algebraic topology. Cambridge: Cambridge University Press. p. 185